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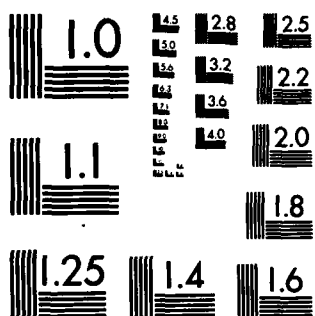NATIONAL BUREAU OF STANDARDS-1963-A

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SENSOR CORRELATION AND DATA FUSION THEORY

FINAL REPORT

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ALPHATECH,
INC.
2 BURLINGTON EXECUTIVE CENTER
111 MIDDLESEX TURNPIKE
BURLINGTON, MA 01803
617-273-3388

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This report contains a summary of the results of the first two years, and the results obtained during the third year of research in the mathematical problems associated with the analysis and design of Air Force sensor correlation and data fusion systems. These systems play a vital role in the command and control process, but presently there exists no systematic and quantitative methodology for their analysis and design. In the first year of research, ALPHATECH investigated an important subproblem: the distributed detection problem associated with determining the presence or absence of targets from a collection of distributed sensors. In the second year of research, ALPHATECH obtained novel exact expressions for the probability density function of the local log likelihood ratios used in distributed detection problems, and has used these expressions to generate an extensive set of design curves. In the third year of research, ALPHATECH investigated another important class of problems, namely, sequential distributed detection problems. Three such problems were formulated: (1) the infinite horizon decentralized Wald (CONT.)

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ITEM #19, ABSTRACT, CONTINUED: problem; (2) a decentralized Quickest Detection problem; and (3) a sequential distributed detection problem with communication and ordered stopping times. The qualitative properties of the optimal solution for the decentralized Wald and the decentralized Quickest Detection problem were investigated, and simple suboptimal algorithms for the decentralized Wald problem and the sequential distributed detection problem with communication were obtained.

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ALPHATECH Inc.
2 Burlington Executive Center
111 Middlesex Turnpike
Burlington, Massachusetts 01803
(617) 273-3388

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Dr. Joseph Bram
Air Force Office of Scientific Research
Bolling Air Force Base
Washington, DC 20332

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ALPHATECH, INC.

PROJECT PARTICIPANTS

Dr. Nils R. Sandell, Jr.

Dr. Demosthenis Teneketzis

Dr. Gregory S. Lauer

Mr. Steven M. Basile

PROJECT CONSULTANTS

Prof. Alan S. Willsky, Massachusetts Institute of Technology

Prof. Pravin P. Varaiya, University of California at Berkeley

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<div align="center">CONTENTS</div>

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SECTION 1

INTRODUCTION

The classical theory of optimal sensor signal processing is based on statistical estimation and hypothesis testing methods [1]. The salient features of classical signal processing theory is that all sensor signals are implicitly assumed to be available in one place for processing. In recent years, however, there has been an increasing interest in distributed sensor systems. This interest has been sparked by large-scale systems such as power systems, surveillance systems, etc., where because of considerations such as cost, reliability, survivability, communication bandwidth, compartmentalization, or even problems caused by flooding a central processor with more information than it can process, there is never centralization of information in practice. Thus, extensions are needed to the classical framework of detection and estimation theory if it is to be relevant to the design of distributed sensor systems. Such extensions need to take into account (besides the decentralization of information) issues like the timeliness as well as the accuracy of the decisions of the detectors. To illustrate these issues, consider the following example from surveillance systems.

Consider two airplanes flying over a surveillance area and having to decide by some finite time T whether or not there are targets in that area. The airplanes do not communicate so that they are not easily detected. The

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longer the airplanes take measurements the better is the quality of information they collect and the more reliable are their decisions. On the other hand, the longer the airplanes remain over the surveillance area the bigger are the chances that they will be detected and the longer is the time they cannot be used for some other purpose. Thus, accumulation of information is costly and there is a tradeoff between the quality of information upon which the decisions of the airplanes are based and the timeliness of their decisions.

Such an example captures a lot of the basic features of distributed detection problems. The airplanes are the two sensors each one with its own information. The presence or nonpresence of targets in the area can be represented by two hypotheses h (h = 0,1). The final assessment of the two airplanes, as to whether targets are present or not, can be represented by their decisions $u_i(u_i = 0,1 \quad i = 1,2)$. Since information is costly we assume that each measurement (after the initial one) taken by each airplane costs c. The fact that the airplanes have the same goal (i.e. detect targets) can be represented by a common terminal cost $J(u_1, u_2, h)$ which couples their decisions. The overall objective is to minimize the cost due to the measurements and the penalty due to errors in detection. Thus, there is a tradeoff between the quality of information upon which the decisions u_i are based and the timeliness of these decisions. The requirement that the airplanes do not communicate can be incorporated as an information constraint on the decision problem.

The example described above is one of the simplest distributed detection problems; it can be formulated as a decentralized Wald problem and serves as the starting point of a detection theory for distributed sensors described in this report.

SECTION 2

OVERVIEW OF PREVIOUS RESEARCH

The problem of constructing decentralized estimation and hypothesis test-
ing rules can be viewed in the framework of decentralized stochastic optimal
control. Decentralized stochastic control problems have been studied over
the past 15 years (see [2] and references therein). The investigation of
these problems has shown that they are in general very complicated. Even the
classes of decentralized stochastic control problems which are the easiest to
analyze (such as LQG static team problems [3], or team problems with partially
nested information structure [4]) have solutions which are considerably more
complicated than the solution of the corresponding centralized stochastic
control problems. This characteristic carries over to decentralized estima-
tion and detection problems.

There are, however, static or quasistatic decentralized estimation and detec-
tion problems for which the solution has been analytically derived. Tenney
and Sandell [5] considered the first simple distributed detection problem
where there are two hypotheses, denoted 0 or 1, and two detectors. In their
formulation, Tenney and Sandell assumed that the detectors have a common ob-
jective (i.e., their detection problems are coupled through the cost) and each
detector takes one measurement (or a set of measurements) and makes a decision
based on his own information. The measurements of the detectors are assumed
to be independent conditioned on the hypothesis. Under these assumptions it

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was shown ([5]) that the team optimal strategies of the two detectors are described by thresholds which are determined by the solution of two coupled nonlinear algebraic equations.

Lauer and Sandell [6]-[10] extended the results of [5] to the case of correlated waveform observations. They found that in general the determination of the optimal decision rules of the two detectors requires the solution of two coupled nonlinear functional equations. Then, Lauer and Sandell examined several special cases and suboptimal approaches. For the special case of detecting linearly dependent signals in white noise, they determined that the local likelihood ratio is a sufficient statistic for detection and they computed numerical examples for the case where the signal is a random process. They examined a suboptimal solution consisting of local likelihood ratio tests with jointly optimized thresholds, and obtained results for a number of interesting cases.

Ekchian [11] considered a problem similar to that of Tenney and Sandell [5] but assumed in addition that a unidirectional communication link exists between the two detectors. Ekchian found that the team optimal decision rules of the two detectors are described by thresholds, the computation of which is coupled. In addition, he found that the detector receiving the communication uses one of two thresholds depending on the decision of the other detector.

The work on distributed detection reported in [5]-[11] assumes a model with static hypotheses (i.e., the true hypothesis does not change with time) and static observations (i.e., the detectors take one measurement or a set of measurements and make a decision).

Teneketzis [12] considered a distributed detection problem with static hypotheses and dynamic observations (i.e., at each instant of time, each

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detector can either stop and make a decision or request more information at some cost). Teneketzis [12] formulated a finite horizon decentralized optimal stopping problem with two hypotheses and two detectors, which is the decentralized version of Wald's problem. He found that the optimal decision rules of the two detectors are described by thresholds. The thresholds of the two detectors are time varying and coupled, and are determined by the solution of $4N-2$ nonlinear algebraic equations in $4N-2$ unknowns, where N is the horizon of the problem.

Subsequently, Teneketzis and Varaiya [13] solved a distributed detection problem with dynamic events (i.e., the case in which the true hypothesis changes with time) and dynamic observations. They formulated an infinite horizon decentralized optimal stopping problem, with two hypotheses and detectors, which is the decentralized version of a quickest detection proJ . They found that the optimal decision rules of the two detectors are described by time-varying thresholds which can be determined by the solution of two coupled dynamic programming equations.

Kushner and Pacut [14] studied a decentralized detection and coordination problem via simulation. They considered two hypotheses, 0 or 1, and two detectors. Each detector takes an observation at time 1 and may, if it wishes, take an observation at time 2. The second observation costs C. The detectors do not communicate with each other. At the end of its "observation period" each detector transmits its conditional probabilities of the hypotheses to a coordinator who then computes the posterior probability and decides on either 0 or 1. Kushner and Pacut [14] investigated the effects of prior probabilitiy and parametric dependencies on the decision rules, as well as sensitivity to the data, asymmetries in the design rules and other phenomena.

The problems studied in [5]-[14] are decentralized detection problems.

Barta, [15], first formulated a decentralized estimation problem. He considered two agents (estimators) who have access to different information, do not communicate, and want to estimate a Gaussian random variable. The estimation rules of the agents were coupled through the cost function. Barta restricted attention to linear estimation rules and derived a recursive solution for the best linear estimates of the agents.

The objective of the problems studied in [5]-[15] was the determination of the optimal decision rules of the agents in distributed estimation and detection problems.

Borkar-Varaiya [16], Tsitsiklis-Athans [17], Washburn-Teneketzis [18] and Teneketzis-Varaiya [19] considered a different class of distributed estimation and detection problems. Namely, [16]-[19] consider a set of distributed communicating agents; each agent makes a decision (estimate, detection, etc.) according to a fixed prespecified rule, and communicates it to the other agents who update their decisions according to the same rule. The convergence and agreement of the decisions (estimates, detections, etc.) is the topic of investigation in [16]-[19].

Borkar and Varaiya [16] investigated the consensus problem when the agents generate and exchange the conditional expectations of the random variable they want to estimate. They showed that the agents' estimates eventually agree. Tsitsiklis and Athans [17] extended the results of [16] to the case where the objective function is an arbitrary convex function of the decision variables of the agents. They showed that when the decisions generated by the optimal decision rule are communicated, then asymptotically the decisions of the distributed agents agree.

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Washburn and Teneketzis [18] considered a setup similar to that of [16] and [17] and characterized the properties which any decision rule must satisfy in order to result in agreement among the communicating agents.

The results reported in [16]-[18] assume that all the communicating agents have the same probabilistic description of the random variables they attempt to estimate or detect.

Teneketzis and Varaiya [19] considered the consensus problem for a distributed estimation problem where the agents have a different probabilistic description of the random variables they want to estimate and where they exchange their conditioned expectations; they demonstrated the effect of different models in the outcome of the estimation process.

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SECTION 3

SUMMARY OF RESEARCH RESULTS

During the first two years of our research we concentrated on distributed detection problems with static hypotheses and static observations. The main features of these problems are: There are two or more detectors and two hypotheses; the true hypothesis does not change with time; the detectors have access to different information, do not communicate, they take one observation and make a decision; the decisions of the detectors are coupled through their common objective.

The results of our research on distributed detection with static hypothesis and static observations have been documented in [6]-[10]. As pointed out in Section 2, Lauer and Sandell ([6]-[10]) treat distributed detection problems with correlated waveform observations, and examine optimal and suboptimal approaches for the solution of these problems.

During the third year of our research we concentrated on distributed sequential detection problems. The main features of these problems are: There are two or more detectors who have access to different information and have to detect one of two possible events as quickly and as accurately as possible. The detectors may or may not communicate. If they communicate they can only exchange limited information.

We have formulated and studied three distributed sequential detection problems: (1) The Infinite Horizon Decentralized Wald Problem, (2) An

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Extension of the Decentralized Quickest Detection Problem and (3) A distributed sequential detection problem with limited communication and ordered stopping times.

For the infinite horizon decentralized Wald problem ([20]) we derived the qualitative properties of the optimal decision rules of the detectors. We showed that the team optimal decision rules of the detectors are characterized by time invariant thresholds whose computation requires the solution of two coupled sets of dynamic programming equations. Since the numerical solution of those equations is difficult, we developed a suboptimal algorithm which captures the basic features of the optimal solution, and it is easy to implement. The study of the qualitative properties of the optimal solution of the infinite horizon decentralized Wald problem appears in Appendix B. The description of the suboptimal algorithm as well as the numerical results derived from its implementation appear in Appendix C.

The decentralized Quickest Detection Problem ([21]) studied during this year is an extension of [13]. We consider two detectors who take independent noisy observations of a two-state Markov chain and have to decide when the chain jumps from state 0 to state 1. The decisions of the detectors are coupled through a common cost function where delays in detection of the jump as well as false alarms are linearly penalized. It is shown that the optimal decision rules of the detectors are characterized by thresholds. These thresholds are time-varying and their computation requires the solution of two coupled sets of dynamic programming equations. Numerical solution of these equations is very difficult; thus, the paper provides only a qualitative characterization of the optimal decision rules of the detectors. However, a comparison with the thresholds of a class of centralized finite horizon quickest

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detection problems is possible. Such a comparison reveals the nature of the coupling. The results of our study on the Quickest Detection Problem are documented in Appendix A ([21]).

Distributed sequential detection problems without any communication among the detectors are in general team problems with static information structure. When the detectors are allowed to communicate and exchange limited information, then this limited communication results in dynamic team problems which are not sequentially decomposable. It is very difficult to analyze these problems, that is why good suboptimal solutions are necessary.

We considered a simple distributed sequential detection problem with limited communiction and ordered stopping times; and we developed a suboptimal algorithm which is simple to implement. The proposed algorithm is very similar to the one proposed for the infinite horizon decentralized Wald problem; the thresholds used are the same as in the suboptimal algorithm for the Wald problem; a simple suboptimal interpretation of the communicating message is suggested; such an interpretation avoids the difficulties introduced by signaling. Further testing of the proposed suboptimal algorithm is needed. The formulation of the distributed sequential detection problem with communication as well as the suboptimal algorithm proposed for its solution are documented in Appendix C ([22]).

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APPENDIX A

TECHNICAL PAPER 192

AN EXTENSION OF THE DECENTRALIZED QUICKEST DETECTION PROBLEM

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TECHNICAL PAPER 192

AN EXTENSION OF THE DECENTRALIZED
QUICKEST DETECTION PROBLEM

By

Demosthenis Teneketzis

Nils R. Sandell, Jr.

ALPHATECH, Inc.
2 Burlington Executive Center
111 Middlesex Turnpike
Burlington, Massachusetts 01803

ABSTRACT

Two detectors take independent noisy observations of a two-state ($\{0,1\}$) Markov chain and have to decide when the Markov chain jumps from state 0 to state 1. The decisions are coupled through a common cost function where delays in detection of the jump as well as false alarms are linearly penalized. It is shown that the optimal decision rules of the detectors are characterized by thresholds. These thresholds are time-varying and their computation requires the solution of two coupled sets of dynamic programming equations. Numerical solution of these equations is not possible; thus the paper provides only a qualitative characterization of the optimal decision rules of the detectors. However, a comparison with the thresholds of a class of centralized finite horizon quickest detection problems is possible. Such a comparison shows the structure of the coupling.

Key Words: Decentralized Quickest Detection, Markov chain, Stopping Rules, Dynamic Programming, Threshold.

1. INTRODUCTION

Two detectors make independent observations of a Markov chain x_t which jumps from state 0 to state 1 at some random time θ. Each detector has to detect the time of the jump based on its own noisy measurements; let τ_i be the time detector i declares that the jump has occurred. The problem is to find stopping times τ_i which minimize the expected cost $EJ(\tau_1,\tau_2,\theta)$.

If the cost is separable, i.e., $J(\tau_1,\tau_2,\theta) = J(\tau_1,\theta)+J(\tau_2,\theta)$, then the decisions of the two detectors are decoupled. In such a case for certain costs τ_i, (see [1],[2]) the optimal decision τ_1^* is to stop and declare the jump the first time instant the probability of "false alarm" drops below a time-invariant threshold ℓ_i, i.e.,

$$\tau_1^* = \min\left\{t \,\middle|\, \text{Prob}(\theta>t|Y_t^i) < \ell_i\right\} \qquad (1-1)$$

where Y_t^i is the information available to detector i at time t. The threshold property holds for the cost functions

$$J(\tau,\theta) = C(\tau,\theta)\, 1(\tau>\theta) + 1(\tau<\theta) \qquad (1-2)$$

and

$$J(\tau,\theta) = C(\tau-\theta)\, 1(\tau>\theta) + k(\theta-\tau)\, 1(\tau<\theta) \qquad (1-3)$$

If the cost $J(\tau_1,\tau_2,\theta)$ is not separable, then there is an interaction between the optimal decisions. Detection problems with nonseparable costs have been previously investigated in [3]-[8]. The problem investigated in

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this paper has a nonseparable cost but is essentially different from those of [5]-[8] which are not sequential; it is also different from the problem of [4] where the Markov chain is frozen in one of two states; it is similar to the problem considered in [3[, but is has a cost function different from that of [3]. The cost function considered in [3] was

$$J(\tau_1, \tau_2, \theta) = C(\tau_1 - \theta) \, 1(\tau_1 > \theta) + C(\tau_2 - \theta) \, 1(\tau_2 > \theta) + 1(\tau_1 < \theta) \, 1(\tau_2 < \theta)$$

(1-4)

This cost puts a constant penalty for false alarms and, for each detector, a penalty proportional to the delay in detecting the jump.

The cost function considered in this paper is

$$J(\tau_1, \tau_2, \theta) = k(\theta - \tau_1 + \theta - \tau_2) \, 1(\tau_1 < \theta) \, 1(\tau_2 < \theta) + C(\tau_1 - \theta) \, 1(\tau_1 > \theta) + C(\tau_2 - \theta) \, 1(\tau_2 > \theta)$$

(1-5)

This cost puts, for each detector, a penalty proportional to the delay in detecting the jump and a penalty proportional to how early false alarms occurred. Thus, the coupling of the detectors (described by the term $k(\theta - \tau_1 + \theta - \tau_2) \, 1(\tau_1 < \theta) \, 1(\tau_2 < \theta)$) is different from that of [3]. Because of the coupling through the cost function, there is an interaction between the optimal decisions of the detectors. The interaction is simple since there is no communication between the detectors.

In this paper we show that the member by member optimal (m.b.m.o.) decisions of the two detectors are described by thresholds as in [3]. These thresholds are time varying, as in [3], and their computation requires the solution of two coupled sets of dynamic programming equations. It is not possible to solve numerically these equations, thus the paper provides only

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a qualitative characterization of the optimal decision rules of the detectors. However, it is possible to compare the m.b.m.o thresholds with the thresholds of a centralized problem. We prove that for each instant of time t the m.b.m.o. thresholds lie above the thresholds at t of class of finite horizon N (N>t) centralized quickest detection problems; the thresholds of these centralized problems are time-varying. A similar comparison was achieved in [3]. However, whereas in [3] the m.b.m.o. thresholds were compared to a stationary threshold, such a comparison is impossible for the present problem.

The remainder of the paper is organized as follows: The formal model is presented in Section 2. The characterization of the m.b.m.o. solutions of the decentralized quickest detection problem is presented in Section 3. In Section 4 the comparison of the m.b.m.o. thresholds with the thresholds of a class of finite horizon centralized quickest detection problems is made. Concluding remarks appear in Section 5.

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2. THE MODEL

Consider of Markov chain $\{x_t, t=1,2,\ldots\}$ with values in $\{0,1\}$, known transition probabilities

$$\text{Prob}(x_{t+1} = 1 | x_t = 0) = p \qquad (2\text{-}1)$$

$$\text{Prob}(x_{t+1} = 1 | x_t = 1) = 1 \qquad (2\text{-}2)$$

and

$$\text{Prob}(x_1 = 0) = 1 \qquad (2\text{-}3)$$

Thus, the chain makes a jump to state 1 at the random time $\theta \underline{\Delta} \min\{t : x_t = 1\}$. Detector i's observation at time t is

$$y_t^i = g^i\left(x_t, w_t^i\right) \qquad i = 1,2 \qquad (2\text{-}4)$$

where it is assumed that $\{w_t^i\}$, $i=1,2$, are mutually independent i.i.d. sequences which are also independent of $\{x_t\}$.

The problem the detectors are faced with is the following:

$$\left.\begin{array}{l} \underset{\tau_1}{\text{Min}} \ \underset{\tau_2}{\text{Min}} \ E\{k\left(\theta - \tau_1 + \theta - \tau_2\right) 1\left(x_{\tau_1} = 0\right) 1\left(x_{\tau_2} = 0\right) \\[2ex] + C \sum_{t=1}^{\tau_1 - 1} 1\left(x_t = 1\right) + C \sum_{t=1}^{\tau_2 - 1} 1\left(x_t = 1\right)\} \qquad (2\text{-}5) \end{array}\right\} P$$

subject to Eqs. 2-1 through 2-4.

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where τ_i is a Y^i_t stopping time and

$$Y^i_t \triangleq \sigma\left(y^i_s, \ s < t\right) \tag{2-6}$$

Problem (P) is a team problem. In this paper we shall derive certain qualitative properties of the solution of problem (P). We shall prove that the member by member optimal stopping rules of the detectors are characertized by time varying thresholds whose computation requires the solution of two coupled sets of dynamic programming equations.

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3. CHARACTERIZATION OF THE OPTIMAL SOLUTION

In this section we shall characterize the member by member optimum solutions of problem (P). The main result of the paper which we shall prove in this section can be summarized by the following theorem:

Theorem 3.1

The member-by-member optimum decision rules of the detectors are characterized by thresholds as in the case of a single detector. However, the thresholds are time-varying and their computation requires the solution of two coupled sets of dynamic programming equations.

The proof of this theorem proceeds in various steps.

Fix $\tau_2 \underline{\Delta} \tau_2^*$ (possibly at the optimum). Then, the problem faced by detector i is

$$\operatorname*{Min}_{\tau_1} EJ(\tau_1) = \operatorname*{Min}_{\tau_1} E \left\{ k\left(\theta-\tau_1 + \theta-\tau_2^*\right) 1\left(x_{\tau_2^*}=0\right) 1\left(x_{\tau_1}=0\right) \right.$$
$$\left. + C \sum_{t=1}^{\tau_1-1} 1\left(x_t=1\right) \right\} \qquad (3-1)$$

Define

$$\pi_t^1 \underline{\Delta} \operatorname{Prob}\left(x_t=0|Y_t^1\right) = \operatorname{Prob}\left(\theta>t|Y_t^1\right) \qquad (3-2)$$

Then, we can alternatively write the cost in Eq. 3-1 as follows:

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$$EJ(\tau_1) = E\left\{ C \sum_{t=1}^{\tau_1} (1-\pi_t^1) + H_{\tau_1}^1 \, \pi_{\tau_1}^1 \right\} \tag{3-3}$$

where

$$H_t^1 = k \sum_{\ell=1}^{\infty} \ell(1-p)^{\ell-1} \, p \, E\left\{ 1\left(\tau_2^* < \theta\right) \big| \theta = t+\ell \right\}$$

$$+ k \sum_{r=0}^{t} \mathrm{Prob}\left(\tau_2^* < r \big| \theta > r\right) + \sum_{\ell=1}^{\infty} \mathrm{Prob}\left(\tau_2^* < t+\ell \big| \theta > t+\ell\right)(1-p)^{\ell}$$

$$= k \sum_{\ell=1}^{\infty} (1-p)^{\ell-1} \, \mathrm{Prob}\left(\theta > \tau_2^* \big| \theta > t+\ell\right) \tag{3-4}$$

$$+ k\left[\frac{1}{p} + \left(t - E(\tau_2^*)\right)^{\dagger}\right] E\left[(1-p)^{\left(\tau_2^* - t\right)^{\dagger}}\right]$$

and

$$[z]^{\dagger} = \max(0,z) \tag{3-5}$$

The equality of the cost functions in Eqs. 3-1 and 3-3 is shown in Appendix A. Thus, detector 1 has to determine a Y_t^1 stopping time to minimize Eq. 3-3. Since the detectors do not communicate and τ_2 is fixed, a Y_t^1 stopping time for detector 1 can be determined by Dynamic Programming. If H_t^1 were constant then the problem faced by detector 1 for fixed τ_2 would be a quickest detection problem similar to that studied in [1],[2]. Since H_t^1 is time-varying it is necessary to study the value function of the dynamic program.

3.1 DYNAMIC PROGRAMMING

To study the value functions we first consider (as in [3]) a finite horizon problem.

Finite Horizon

Fix the final time T (T<∞) and consider the problem

$$\text{Minimize } EJ(\tau_1) \qquad (3\text{-}6)$$
$$1 < \tau_1 < T$$

The dynamic programming argument for this problem shows that the value function $V_t^{1T}(\pi)$, i.e.,

$$V_t^{1T}(\pi^1) = \min_{t < \tau_1 < T} E\left\{ H_{\tau_1}^1 \, 1(x_{\tau_1}=0) + C \sum_{r=t}^{\tau_1-1} 1(x_r=1) \,\bigg|\, \pi_t^1 = \pi^1 \right\} \qquad (3\text{-}7)$$

is obtained by

$$V_T^{1T}(\pi^1) = H_T^1 \, \pi^1 \qquad (3\text{-}8)$$

$$V_t^{1T}(\pi^1) = \min\{ H_t^1 \, \pi^1 \, , \, (L \, V_{t+1}^{1T})(\pi^1) + C(1-\pi^1) \} \qquad t < T \qquad (3\text{-}9)$$

where

$$(LV)(\pi^1) = \int V(A(\pi^1,y)) \, q(y|\pi^1) \, dy \qquad (3\text{-}10)$$

$$q(y|\pi^1) = \pi^1(1-p) \, P_0^1(y) + \pi^1 \, p \, P_1^1(y) + (1-\pi^1) \, P_1^1(y) \qquad (3\text{-}11)$$

$$A(\pi^1,y) = \pi^1(1-p) \, P_0^1(y)/q(y|p) \qquad (3\text{-}12)$$

and

$P_i^1(y)$ is the probability density of the measurement y_t^1 under the assumption that $x_t=i$. The term $H_t^1 \, \pi^1$ in Eq. 3-9 describes the cost incurred by the decision to stop at time t and the term $C(1-\pi^1) + (L \, V_{t+1}^{1T}) \, (\pi^1)$ describes the cost incurred by the decision at t to continue taking measurements. Thus, it is optimal for detector 1 to stop if and only if

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$$H^1_t \pi^1_t < C(1-\pi^1) + \left(L \; V^{1T}_{t+1}\right)\left(\pi^1_t\right) \qquad (3\text{-}13)$$

The value function $V^{1T}_t (\pi)$ has the following important property:

Lemma 3.1

$V^{1T}_t (\pi)$ is a nonnegative concave function of $\pi(t=1,2,\ldots,T)$. $\left(L \; V^{1T}_{t+1}\right)(\pi)$ is also a nonnegative concave function of π. $(t=1,2,\ldots,T)$

Proof: The proof is the same as that of Lemma 3.1 in [3]. □

Lemma 3.2

At $\pi=0$

$$H^1_t \pi < C(1-\pi) + \left(L \; V^{1T}_{t+1}\right) (\pi) \qquad (3\text{-}14)$$

At $\pi=1$

$$H^1_t \pi > C(1-\pi) + \left(L \; V^{1T}_{t+1}\right) (\pi) \qquad (3\text{-}15)$$

Proof: See Appendix B. □

Lemmas 3.1 and 3.2 imply the threshold property of agent 1's optimal policy for fixed (arbitrary) τ_2 and for the finite horizon problem.

Infinite Horizon

To minimize Eq. 3-3 let $t\to\infty$. Since the set of stopping times τ_1 of agent 1 increases as T increases, it follows that

$$V^{1(T+1)}_t (\pi^1) < V^{1T}_t (\pi^1) \qquad (3\text{-}16)$$

and since the value functions $V^{1T}_t (\pi^1)$ are nonnegative, the following limit is well defined:

$$V^1_t (\pi) \;\underline{\Delta}\; \lim_{t\to\infty} V^{1T}_t (\pi) = \inf_T V^{1T}_t (\pi) \qquad (3\text{-}17)$$

We can show that the value function $V^1_t (\pi)$ has the following properties

Lemma 3.3

The value function $V^1_t (\pi)$ satisfies the equation

$$V^1_t (\pi^1) = \min \left\{ H^1_t \, \pi^1, \; C(1-\pi^1) + \left(L \, V^1_{t+1}\right) (\pi^1) \right\} \quad t=1,2,\ldots \qquad (3\text{-}18).$$

$V^1_t (\pi)$ as well as $\left(L \, V^1_{t+1}\right) (\pi)$ are nonnegative and concave functions of π. Moreover, at $\pi=0$ and $\pi=1$, Eqs. 3-14 and 3-15 hold respectively.

Proof: Eq. 3-18 is obtained by Eq. 3-9. The nonnegativity and concavity of $V^1_t (\pi)$ and $\left(L \, V^1_{t+1}\right) (\pi)$ follow from Lemma 3.1. The inequalities at $\pi=0$ and $\pi=1$ follow from Lemma 3.2. $\square$

Furthermore, it is possible to show that the value functions have the following additional property.

Lemma 3.4

The value functions $\left\{V^1_t (\pi)\right\}$ are the unique solution of

$$Z_t(\pi) = \min \left\{ H^1_t \, \pi, \; C(1-\pi) + \left(L \, Z_{t+1}\right) (\pi) \right\} \quad , \quad t=1,2,\ldots \qquad (3\text{-}19)$$

Proof: See [3], Theorem 3.2. $\square$

The properties of the value function $V^1_t (\pi)$ imply the following characterization of agent 1's optimal policy for fixed τ_2.

Lemma 3.5

For fixed τ_2 the optimal stopping time for detector 1 is

$$\tau^*_1 = \min \left\{ t: \quad \pi^1_t < \ell^1_t \right\} \qquad (3\text{-}20)$$

The threshold ℓ_t^1 is defined by the solution of:

$$H_t^1 \ \pi = C(1-\pi) + \left(L \ V_{t+1}^1\right)(\pi) \qquad (3-21)$$

<u>Proof</u>: The concavity of $\left(L \ V_{t+1}^1\right)(\pi)$ and the inequalities 3-14 and 3-15 imply imply that the functions $H_t^1 \ \pi$ and $C(1-\pi) + \left(L \ V_{t+1}^1\right)(\pi)$ intersect at one point. The solution of Eq. 3-21 defines this point. From Eq. 3-18 the optimal decision optimal decision rule of agent 1 at time t is to stop if and only if

$$V_t^1 \ (\pi) = H_t^1 \ \pi \qquad (3-22)$$

Equation 3-22 gives the rule (3-20) QED. $\square$

We are now in a position to prove the main result of this paper which was summarized by Theorem 3.1.

Proof of Theorem 3.1

Since the optimum policy of agent 1 is characterized by thresholds for any arbitrary fixed policy of agent 2, it will also be characterized by thresholds when the policy of agent 2 is the optimum. By symmetry, the optimal policy of agent 2 is characterized by thresholds. Hence, the member-by-member optimal policies of the agents are characterized by thresholds $\left\{\ell_t^{1*}, \ell_t^{2*}\right\}$. These thresholds are determined by the solution of a set of coupled dynamic programming equations which are of the form

$$H_t^1\left(\left\{\ell_t^{2*}\right\}\right) \ \ell_t^{1*} = C\left(1-\ell_t^{1*}\right) + \left(L \ V_{t+1}^1\right)\left(\ell_t^{1*}\right) \qquad t=1,2,\ldots \qquad (3-23)$$

$$H_2^\varsigma\left(\left\{\ell_t^{1*}\right\}\right) \ \ell_t^{2*} = C\left(1-\ell_t^{2*}\right) + \left(L \ V_{t+1}^1\right)\left(\ell_t^{2*}\right) \qquad t=1,2,\ldots \qquad (3-24)$$

The proof of Theorem 3.1 is now complete. $\square$

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It is not possible to solve numerically the set of coupled dynamic programming
equations described by Eqs. 3-23 and 3-24; thus, the results of this paper
provide only a qualitative characterization of the optimal decision rules of
the detectors. It is possible, however, to compare the thresholds $\left\{ \ell_t^{1*} , \ell_t^{2*} \right\}$
with the thresholds of centralized quickest detection problems as shown in
the next section.

4. COMPARISON WITH A CLASS OF CENTRALIZED QUICKEST DETECTION PROBLEMS

In this section we shown that for each time t it is possible to compare the m.b.m.o. thresholds of problem (P) with a class of finite horizon centralized quickest detection problems. The thresholds of these centralized problems are time-varying. Thus, the results obtained in this section are different from those of [3] where comparison of the m.b.m.o. thresholds with a time-invariant thresholds was possible.

Let $\left(\ell_t^{1*}, \ell_t^2\right)$ be two m.b.m.o. thresholds for problem (P) at time t. Consider a finite horizon centralized quickest detection problem [1],[2] with final time $N > t$ and cost

$$J(\tau,\theta) = E\left\{ G_\tau^N \, 1(\tau < \theta) + C \sum_{t=1}^{\tau-1} 1(x_t = 1) \right\} \tag{4-1}$$

where

$$G_t^N = K\left\{ \frac{1}{b} \sum_{\ell=1}^{N-t} \ell(1-p)^{\ell-1} p + N + 2\frac{1}{p} \right\} \tag{4-2}$$

and

$$b = \sum_{\ell=1}^{N} \ell(1-p)^{\ell-1} p \tag{4-3}$$

It is well known, [1],[2], that the optimal stopping rule for the above problem is described by thresholds $\left\{\lambda_t^N\right\}_{t=1}^{N}$ which are determined by the solution of a set of equations in the form

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$$G_t^N \lambda_t^N = C\left(1-\lambda_t^N\right) + \left[L \ M_{t+1}^N\right]\left(\lambda_t^N\right) \tag{4-4}$$

where $M_t^N (\pi)$ is defined by

$$M_N^N (\pi) = k\left(N + 2\frac{1}{p}\right)\pi \tag{4-5}$$

$$M_t^N (\pi) = \min\left\{G_t^N \pi \ , \ C(1-\pi) + \left[L \ M_{t+1}^N\right](\pi)\right\} \quad t=1,2,\ldots,N-1 \tag{4-6}$$

and $\left[L \ M_{t+1}^N\right] (\pi)$ is defined in a way analogous to Eqs. 3-10 through 3-12.

The main result of this section is summarized by the following theorem:

Theorem 4.1

Let $\left(\ell_t^{1*} , \ell_t^{2*}\right)$ be two m.b.m.o. thresholds for problem (P) at time t, and let λ_t^N be the optimal threshold at time t for the centralized quickest detection problem with final time N and cost given by Eq. 4-1. Then, for all $t < N$

$$\lambda_t^N < \ell_t^{i*} \quad i=1,2 \ . \tag{4-7}$$

The proof of the theorem proceeds in various steps. First, we prove the following lemmas:

Lemma 4.1

For any $t,s \ , \ t>s$

$$G_t^N < G_s^N \tag{4-8}$$

Proof: Follows directly from the definition of G_t^N. □

Lemma 4.2

For any t,s , $t>s$

$$H^i_t > H^i_s \quad (i=0,2) \tag{4-9}$$

where H^i_t is defined by Eq. 3-4.

Proof: See Appendix C.

Lemma 4.3

For all $t<N$,

$$H^i_t < G^N_t , \quad (i=1,2) \ . \tag{4-10}$$

Proof: Follows directly from the definitions of H^i_t and G^N_t.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1

It suffices to prove that for any stopping rule τ^f_2 the corresponding threshold of ℓ^1_t satisfies.

$$\lambda^N_t < \ell^1_t \tag{4-11}$$

From Lemma 3.5, $\pi > \ell^1_t$ if and only if

$$\pi > \left(H^1_t\right)^{-1} \min_{\tau>t} E \left\{ H^1_\tau \ 1(x_\tau=0) + C \sum_{s=t}^{\tau-1} 1(x_s=1) \Big| \pi_t=\pi \right\} \tag{4-12}$$

Because of Lemmas 4.1 through 4.3, Eq. 4-12 gives

$$\pi > \min_{t<\tau<N} E \left\{ \frac{G^N_\tau}{G^N_t} \ 1(x_\tau=0) + \left(G^N_t\right)^{-1} C \sum_{s=t}^{\tau-1} 1(s_x=1) \Big| \pi_t=\pi \right\} \tag{4-13}$$

Equation 4-13 implies that

$$G^N_t \pi > M^N_t(\pi) \qquad (4-14)$$

Consequently

$$\ell^1_t > \lambda^N_t \qquad (4-15)$$

$$\text{QED.}$$

The result of this theorem is similar to that of [3]. However, whereas in [3] comparison with a stationary threshold of a centralized problem is possible, such a comparison is impossible for problem (P). Intuitively, this can be explained as follows: For the problem considered in [3] at each instant of time each detector has to account for an average penalty due to a false alarm of the other detector. Such a penalty varies with time, however, it remains bounded for all times (actually it is less than one). This feature makes possible the comparison of the decentralized m.b.m.o. thresholds with a stationary centralized threshold. For the problem considered in this paper, the average cost faced by each detector is not necessarily bounded. This feature of the cost makes the comparison of thresholds a more subtle problem; comparison with a centralized threshold is only possible if that threshold corresponds to a cost which has similar features as the average cost due to a false alarm in problem (P). This can be accomplished if one restricts attention to finite horizon centralized problems which result in time-varying thresholds.

5. CONCLUDING REMARKS

The problem considered in this paper is similar to those of [3]-[8] in that there is no communication between the two detectors. Even though the problem treated here is one of the simplest decentralized sequential detection problems, the coupling induced by the cost structure causes, (as in [3] and [4]), considerable complexity in the optimal stopping rules. Computation of the optimal thresholds is impossible as it requires the solution of two sets of coupled dynamic programming equations. It is possible, however, to lower-bound the optimal decentralized thresholds by the thresholds of certain finite horizon centralized quickest detection problems. The thresholds of these centralized problems are time-varying. A similar comparison was achieved in [3]. However, whereas in [3] the optimal decentralized thresholds are compared to a stationary centralized thresholds, such a comparison is impossible for the present problem.

The results of this paper provide only a qualitative characterization of the optimal decision rules of the detectors. The qualitative properties of the optimal decision rules as well as the comparison between the optimal decentralized thresholds and a class of optimal centralized thresholds provide the basis for the development of practical algorithms for the problem studied in this paper.

APPENDIX A

Proof of the Equality of Cost Functions (3.1) and (3.3)

Consider first the term

$$E\left\{ C \sum_{\ell=1}^{t} 1(x_\ell=1) \right\} \tag{A-1}$$

According to [1, pp. 151-152], Eq. A-1 is equal to

$$E\left\{ C \sum_{\ell=1}^{t-1} \left(1-\pi_\ell^1\right) \right\} \tag{A-2}$$

Next consider the term $E\left\{k(\theta-t) \ 1(t<\theta) \ 1\left(\tau_2^*<\theta\right)\right\}$.

$$E\left\{k(\theta-t) \ 1(t<\theta) \ 1\left(\tau_2^*<\theta\right)|Y^{1t}\right\} = E\left\{k(\theta-t) \ 1(t<\theta) \ h_1(\theta)|Y^{1t}\right\} \tag{A-3}$$

where

$$h_1(\theta) = E\left\{1\left(\tau_2^*<\theta\right)|\theta\right\} \tag{A-4}$$

The equality of Eq. A-3 holds because the observations of the detectors are independent conditioned on the time of the jump.

We can write

$$E\left\{k(\theta-t) \ 1(t<\theta) \ h_1(\theta)|y^{1t}\right\} = k \sum_{\ell=t+1}^{\infty} h_1(\ell) \ (\ell-t) \ \text{Prob}(\theta=\ell|Y^{1t})$$

$$= k \sum_{\ell=1}^{\infty} h_1(t+\ell)\ell \ \text{Prob}(\theta=t+\ell|Y^{1t}) \tag{A-5}$$

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But

$$\text{Prob}\left(\theta = t + \ell \mid Y^{1t}\right) = \text{Prob}\left(\theta > t \mid Y^{1t}\right)\,\text{Prob}\left(\theta = t + \ell \mid x_t = 0\right) = \pi_t^1\,(1-p)^{\ell-1}\,p$$

(A-6)

Consequently, Eqs. A-3 and A-5 give

$$E\left\{k(\theta - t)\,1(t < \theta)\,1(\tau_2^* < \theta) \mid Y^{1t}\right\} = k\sum_{\ell=1}^{\infty} \ell(1-p)^{\ell-1}\,p\,E\left\{1(\tau_2^* < \theta) \mid \theta = t + \ell\right\}\,\pi_t^1$$

(A-7)

Using the equality

$$\text{Prob}\left(\tau_2^* < \theta \mid \theta > t\right) = \sum_{\ell=1}^{\infty} \text{Prob}\left(\tau_2^* < \theta \mid \theta = t + \ell\right)(1-p)^{\ell-1}\,p$$

(A-8)

we can also write Eq. A-7 as

$$E\left\{k(\theta - t)\,1(t < \theta)\,1(\tau_2^* < \theta) \mid Y^{1t}\right\} = k\sum_{\ell=1}^{\infty}(1-p)^{\ell-1}\,\text{Prob}\left(\theta > \tau_2^* \mid \theta > t + \ell\right)$$

(A-9)

Finally, consider the term $E\left\{k(\theta - \tau_2^*)\,1(\tau_2 < \theta)\,1(t < \theta)\right\}$.

$$k\,E\left\{(\theta - \tau_2^*)\,1(\tau_2^* < \theta)\,1(t < \theta) \mid Y^{1t}\right\} = k\,E\left\{1(t < \theta)\,h_2(\theta) \mid Y^{1t}\right\}$$

(A-10)

where

$$h_2(\theta) = E\left\{(\theta - \tau_2^*)\,1(\tau_2^* < \theta) \mid \theta\right\}$$

(A-11)

Equality in Eq. A-11 holds because the observations of the detectors are independent conditioned on the time of jump.

Using Eqs. A-10 and A-6 we can write

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$$E\left\{(\theta-\tau_2^*)\ 1\left(\tau_2^*<\theta\right)\ 1\left(t<\theta\right)\Big|Y^{1t}\right\} = \sum_{\ell=1}^{\infty} E\left\{(\theta-\tau_2^*)\ 1\left(\tau_2^*<\theta\right)\Big|\theta=t+\ell\right\}\ (1-p)^{\ell-1}\ p\ \pi_t^1$$

$$= \sum_{\ell=1}^{\infty}\left\{\sum_{k=0}^{t+\ell-1}(t+\ell-k)\ \mathrm{Prob}\left(\tau_2^*=k\Big|\theta=t+\ell\right)\right\}\ (1-p)^{\ell-1}\ p\ \pi_t^1$$

$$= \sum_{\ell=1}^{\infty}\left\{\sum_{k=0}^{t+\ell-1}\mathrm{Prob}\left(\tau_2^*<k\Big|\theta=t+\ell\right)\right\}\ (1-p)^{\ell-1}\ p\ \pi_t^1$$

$$= \left[\sum_{r=0}^{t}\mathrm{Prob}\left(\tau_2^*<r\Big|\theta>t\right) + \sum_{\ell=1}^{\infty}\mathrm{Prob}\left(\tau_2^*<t+\ell\Big|\theta>t+\ell\right)(1-p)^{\ell}\right]\ \pi_t^1$$

<div align="right">(A-12)</div>

Note that when r<t

$$\mathrm{Prob}\left(\tau_2^*<r\Big|\theta>t\right) = \mathrm{Prob}\left(\tau_2^*<r\Big|\theta>r\right) \qquad (A-13)$$

Consequently, Eqs. A-12 and A-13 give

$$k\ E\left\{(\theta-\tau_2^*)\ 1\left(\tau_2^*<\theta\right)\ 1\left(t<\theta\right)\Big|Y^{1t}\right\} = \left[\sum_{r=0}^{t}\mathrm{Prob}\left(\tau_2^*<r\Big|\theta>r\right)\right.$$

<div align="right">(A-14)</div>

$$\left. + \sum_{\ell=1}^{\infty}\mathrm{Prob}\left(\tau_2^*<t+\ell\Big|\theta>t+\ell\right)(1-p)^{\ell}\right]\ \pi_t^1$$

An alternative expression for $kE\left\{(\theta-\tau_2^*)\ 1\left(\tau_2^*<\theta\right)\ 1\left(t<\theta\right)|Y^{it}\right\}$ can be derived as follows:

Using Eqs. A-6 and A-10 we can write

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$$k \ E \left\{ (\theta - \tau_2^*) \ 1\left(\tau_2^* < \theta\right) \ 1(t < \theta) \Big| Y^1 t \right\}$$

$$= k \sum_{\ell=1}^{\infty} E \left\{ (\theta - \tau_2^*) \ 1\left(\tau_\ell^* < \theta\right) \Big| \theta = t + \ell \right\} \ \text{Prob}\left(\theta = t + \ell \Big| \theta > s\right) \ \pi_t^1$$

$$= k \ E \left\{ (\ \theta - \tau_2^*) \ 1\left(\tau_2^* < \theta\right) \Big| \theta > t \right\} \ \pi_t^1$$

$$= k \ E \left\{ (\theta - \tau_2^*) \Big| \theta > \tau_2^* \ , \ \theta > t \right\} \ \text{Prob}\left(\theta > \tau_2^*\right) \Big| \theta > t \right) \ \pi_t^1$$

$$= k \ E \left\{ (\theta - \tau_2^*) \Big| \theta > \tau_2^* \ vt \right\} \ \text{Prob}\left(\theta > \tau_2^* \Big| \theta > t\right) \ \pi_t^1$$

$$= k \ E \left\{ (\theta - \tau_2^* \ vt) + \tau_2^* \ vt - \tau_2^* \Big| (\theta - \tau_2^* \ vt) > 0 \right\}$$

$$\times \ \text{Prob} \left\{ (\theta - t) > \tau_2^* - t \Big| \theta - t > 0 \right\} \ \pi_t^1$$

$$= \left[E(\theta) + E\left(\tau_2^* \ vt - \tau_2^*\right) \right] \ \text{Prob}\left(\theta > \tau_2^* - t\right) \ \pi_t^1 \tag{A-15}$$

where

$$\tau_2^* \ vt = \max\left(\tau_2^*, t\right) \tag{A-16}$$

Because of Eqs. 2-1 and 2-2 we can write Eq. A-15 as

$$k \ E \left\{ (\theta - \tau_2^*) \ 1\left(\tau_2^* < \theta\right) \ 1(\tau < \theta) \Big| Y^1 t \right\}$$

$$= k \left[\frac{1}{p} + t - E \left\{ \tau_2^* \ vt - \tau_2^* \right\} \right] E \left\{ (1-p)^{[t_2^* - t]^\dagger} \right\} \ \pi_t^1$$

$$\tag{A-17}$$

$$= k \left[\frac{1}{p} + (t - E(\tau_2^*))^\dagger \right] E \left\{ (1-p)^{[\tau_2^* - t]^\dagger} \right\} \pi_t^1$$

where

$$[x]^\dagger = \max(x, 0) \tag{A-18}$$

Combining Eqs. A-1, A-2, A-7, A-9, A-14, and A-17 we obtain the equality of the cost functions (3.1) and (3.3).

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APPENDIX B

<u>Proof of Lemma 3.2</u>

The inequality (Eq. 3-14) at $\pi=0$ is true because $C>0$ and $\left(L \, V^{1T}_{t+1} \right) (\pi)$ is nonnegative.

To prove Eq. 3-15 note that because of Eqs. 3-9 through 3-12

$$\left(L \, V^{1T}_{t+1} \right) (\pi) < (1-p) \, H^{1T}_{t+1} \, \pi \tag{B-1}$$

Thus, to complete the proof of the lemma, it is enough to show that

$$(1-p) \, H^{1T}_{t+1} < H^{1T}_{t} \tag{B-2}$$

We can easily show, following the arguments in Appendix A, that

$$H^{1T}_{t} = \frac{k}{B} \sum_{\ell=1}^{T-t} (1-p)^{\ell-1} \, \text{Prob}\left(\theta > \tau^*_2 \,\middle|\, \theta > t+\ell\right) + \frac{k}{B} \sum_{\ell=0}^{t} \text{Prob}\left(\tau^*_2 < \ell \,\middle|\, \theta > \ell\right)$$

$$+ \frac{k}{B} \sum_{\ell=1}^{T-t} \text{Prob}\left(\tau^*_2 < t+\ell \,\middle|\, \theta > t+\ell\right)(1-p)^{\ell} \tag{B-3}$$

where

$$B = \sum_{\ell=1}^{T} (1-p)^{\ell-1} \, p \tag{B-4}$$

The inequality B-2 then follows directly from Eqs. B-3 and B-4.   QED

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APPENDIX C

Proof of Lemma 4.2

By definition (Eq. 3-4)

$$H^1_t = k \sum_{\ell=1}^{\infty} (1-p)^{\ell-1} \text{Prob}(\theta > \tau_2^* | \theta > t + \ell)$$

$$+ k \left[\frac{1}{p} + (t - E(\tau_2^*))^\dagger \right] E \left\{ (1-p)^{(\tau_2^*-t)^\dagger} \right\} \quad . \tag{C-1}$$

When $t > s$, then

$$\left[\frac{1}{p} + (t - E(\tau_2^*))^\dagger \right] E \left\{ (1-p)^{(\tau_2^*-t)^\dagger} \right\} >$$

$$\tag{C-2}$$

$$\left[\frac{1}{p} + (s - E(\tau_2^*))^\dagger \right] E \left\{ (1-p)^{(\tau_2^*-s)^\dagger} \right\}$$

It remains to show that

$$\text{Prob}(\theta > \tau_2^* | \theta > t + \ell) > \text{Prob}(\theta > \tau_2^* | \theta > s + \ell) \tag{C-3}$$

for any ℓ, when $t > s$.

But Eq. C-3 is true because of Corollary 4.1 of [3]. Hence $H^1_t > H^1_s$. It can be similarly shown that $H^2_t > H^2_s$ QED.

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APPENDIX B

TR-202

THE INFINITE HORIZON DECENTRALIZED WALD PROBLEM

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TR-202

THE INFINITE HORIZON
DECENTRALIZED WALD PROBLEM

By

D. Teneketzis
N.R. Sandell, Jr.

June 1984

ALPHATECH Inc.
2 Burlington Executive Center
111 Middlesex Turnpike
Burlington, Massachusetts 01803
(617) 273-3388

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ABSTRACT

A Markov chain is frozen in one of two states 0,1. Two detectors making
independent observations must decide what is the state of the Markov chain.
The observations are costly and the decisions of the detectors are coupled
through a common cost function. It is shown that the optimal decision rules
of the detectors are characterized by thresholds whose computation requires
the solution of two coupled sets of dynamic programming equations. Numerical
solution of these equations is very difficult; thus, the results derived in
this report provide only a qualitative characterization of the optimal solu-
tion of the infinite horizon decentralized Wald problem.

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CONTENTS

SECTION 1

INTRODUCTION

In recent years there has been an increasing interest in distributed
sensor systems. This interest has been sparked by large-scale systems such
as power systems, surveillance systems etc., where because of considerations
such as cost, reliability, survivability, communication bandwidth, compart-
mentalization, or even problems caused by flooding a central processor with
more information than it can process, there is never centralization of infor-
mation. Thus distributed estimation and detection problems have recently
received considerable attention as evidenced by [1]-[11]*.

The basic features of the distributed estimation and detection problems
studied so far are: There are more than one agents (detectors, estimators)
which have access to different information and have either to detect an event
([1]-[10]) or estimate a random variable ([11]). The agents' decisions (esti-
mates) are coupled through their common objective. The agents may ([8]) or
may not communicate ([1]-[7], [9]-[11]), but even if they are allowed to com-
municate they cannot exchange their raw data (so that there can be no central-
ization of information). In addition, the agents' decisions may be made at
fixed instants of time ([1]-[4], [8]) or sequentially in time ([5]-[7], [9]).
The results available so far have shown that the distributed information pat-
tern and the coupling of the decisions through the cost introduce considerable

---

*References are indicated by numbers in square brackets, the list appears at
 the end of the main body of this report.

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complexity in the computation of the optimal decision rules. Nevertheless, the same results have revealed a lot of interesting properties of the optimal solutions of distributed detection and estimation problems; these properties can be used to guide the design of simple suboptimal algorithms for these problems.

The goal of the present report is to extend the results of [5] to the infinite horizon decentralized Wald problem. The results derived in this report provide only a qualitative characerization of the optimal solution of the infinite horizon decentralized Wald problem. However, the basic properties of the solution derived here will be used elsewhere, [12], to guide the design of a simple suboptimal algorithm which is easy to implement.

The report is organized as follows: In Section 2 the formulation of the infinite horizon decentralized Wald problem is presented. In Section 3 a brief summary of the basic results of the finite horizon decentralized Wald problem is presented. These results are extended to the infinite horizon problem in Section 4. Conclusions appear in Section 5.

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SECTION 2

PROBLEM FORMULATION

</div>

Consider a two-state $\{0,1\}$ Markov chain h which is "frozen" in one of its states. Assume that

$$\text{Prob}(h=0) = p \qquad\qquad (2\text{-}1)$$

Two agents (detectors) attempt to detect the state h of the chain by taking noisy observations of h. The observations of the two agents are described by

$$y^i(t) = g^i\left(h, \ v^i(t)\right) \quad i = 1,2 \qquad\qquad (2\text{-}2)$$

It is assumed that $\{v^1(t)\}$, $\{v^2(t)\}$ are mutually independent i.i.d. noise sequences which are also independent of h. It is further assumed that each observation after t=1 costs c. Let $u_i(i = 1,2)$ be the final decision of agent i ($u_i = 0$ or 1), and τ_i be the time that decision was made. Then the cost incurred by these decisions is assumed to be

$$c\tau_1 + c\tau_2 + J(u_1, \ u_2, \ h) \qquad\qquad (2\text{-}3)$$

Under these assumptions the infinite horizon decentralized Wald problem is:

<div style="text-align:center">

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</div>

$$\left. \begin{array}{l} \text{Minimize} \quad E\{c\tau_1(\gamma_1) + c\tau_2(\gamma_2) + J(u_1, u_2, h)\} \\ \gamma_1 \epsilon \Gamma_1, \ \gamma_2 \epsilon \Gamma_2 \\[1em] \text{Subject to} \\[1em] u_i \quad \underline{\Delta} \quad \gamma_i(y^{i\tau_i}) \\[1em] y^{i\tau_i} \ \underline{\Delta} \ [y^i(1), \ y^i(2), \ldots, y^i(\tau_i)] \\[1em] \Gamma_i \quad \underline{\Delta} \quad \text{set of stopping rules that are} \\ \qquad\qquad\ \text{measurable functions of the data} \\ \qquad\qquad\ \text{of detector i.} \end{array} \right\} \quad (P)$$

Problem (P) is the infinite horizon version of the problem studied in [5].
In the next section we shall briefly state the main results of [5] and then
we shall extend them to solve problem (P).

SECTION 3

REVIEW OF THE FINITE HORIZON DECENTRALIZED WALD PROBLEM

The finite horizon decentralized Wald problem formulated in [5] is the same as problem (P) with the additional constraint

$$\tau_i < T \quad , \quad i = 1,2 \qquad T < \infty \tag{3-1}$$

i.e., the agents' decisions have to be made no later than T.

It was shown in [5] that the finite horizon decentralized Wald problem is a sequential team problem with static information structure; thus the member-by-member optimal (m.b.m.o.) strategies of the two agents can be determined by dynamic programming methods.

The main results obtained in [5] are the following:

Theorem 3.1

The m.b.m.o. stopping rules of the two agents are described by thresholds α_k^1, β_k^1, α_k^2, β_k^2, ℓ_T^1, ℓ_T^2, $k = 1,2,\ldots,T-1$ (T = horizon of the problem). The thresholds of the two agents are coupled; namely, the thresholds of agent 1(2) at any time t are coupled with the thresholds of detector 2(1) at all times, past, present and future. The computation of these thresholds requires the solution of 4T-2 nonlinear algebraic equations in 4T-2 unknowns. The nonlinear algebraic equations which result from dynamic programming arguments and whose solution determines the thresholds for agent 1 are of the form

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$$\alpha_k^1 \sum_{u_2} p(u_2|h=0)[c\tau_2 + J(1, u_2, 0)]$$

$$+ (1 - \alpha_k^1) \sum_{u_2} p(u_2|h=1)[c\tau_2 + J(1, u_2, 1)]$$

$$= c + E_{y_1(k+1)}\left\{ V_{k+1}^{1T}\left(\frac{\alpha_k^1 \, p(y_1(k+1)|h=0)}{\alpha_k^1 \, p(y_1(k+1)|h=0) + (1-\alpha_k^1) \, p(y_1(k+1)|h=1)} \right) \right\} \tag{3-2}$$

$$\beta_k^1 \sum_{u_2} p(u_2|h=0)[c\tau_2 + J(0, u_2, 0)]$$

$$+ (1-\beta_k^1) \sum_{u_2} p(u_2|h=1)[c\tau_2 + J(0, u_2, 1)]$$

$$= c + E_{y_1(k+1)}\left\{ V_{k+1}^{1T}\left(\frac{\beta_k^1 \, p(y_1(k+1)|h=0)}{\beta_k^1 \, p(y_1(k+1)|h=0) + (1-\beta_k^1) \, p(y_1(k+1)|h=1)} \right) \right\} \tag{3-3}$$

for $k = 1, 2., \ldots, T-1$ and

$$\ell_T^1 = \frac{A(T)}{D(T)} \tag{3-4}$$

$$A(T) = \sum_{u_2} p(u_2|h=1)[J(0, u_1, 1) - J(1, u_2, 1)] \tag{3-5}$$

$$D(T) = A(T) + \sum_{u_2} p(u_2|h=0)[J(1, u_2, 0) - J(0, u_2, 0)] \tag{3-6}$$

In Eqs. 3-2 and 3-3 V_k^{1T} is the value function of the dynamic program for agent 1. Similar equations hold for agent 2.

The conditional probability $\pi_t^i \triangleq p(h=0|y^{it})$ is a sufficient statistic for decisionmaking for agent i (i = 1,2). At any time t (t = 1,2,...,T-1) the action of agent i, based on the statistic π_t^i, can be described as follows:

1. If $\pi^i_t > \beta^i_t$, declare $u_i = 0$ and stop.

2. If $\alpha^i_t < \pi^i_t < \beta^i_t$, continue taking measurements.

3. If $\pi^i_t < \alpha^i_t$, declare $u_i = 1$ and stop.

At time t=T the decision of agent i is:

1. If $\pi^i_t > \ell^i_T$, declare $u_i = 0$

2. If $\pi^i_t < \ell^i_T$, declare $u_i = 1$.

Thus, the stopping times τ_i (i = 1,2,) have the following property:

$$\tau_i = \min \left\{ t: \quad \pi^i_t < \alpha^i_t \quad \text{or} \quad \pi^i_t > \beta^i_t \quad i = 1,2 \quad , \quad 1 < t < T \right\} \quad (3\text{-}7)$$

If there is no t such that Eq. 3-7 holds, then

$$\tau_i = T \quad\quad\quad (3\text{-}8)$$

The coupling of the thresholds of the two agents arises because of the presence of the terms $p(u_2|h=i)$ in Eqs. 3-3 through 3-6; these terms are functions of $\left(\alpha^2_1, \beta^2_1, \ldots, \alpha^2_{T-1}, \beta^2_{T-1}, \ell^2_T \right)$.

The value functions $V^{iT}_k(\pi^i_t)$ have the following properties:

Lemma 3.1

The value function $V^{1T}_t(\pi^1_t)$ is a nonnegative concave function of π^1_t and satisfies the following equations:

$$V^{1T}_T(\pi^1_T) = \min \left\{ \pi^1_T \sum_{u_2} p(u_2|h=0) \, J(0, u_2, 0) \right.$$

$$+ (1-\pi^1_T) \sum_{u_2} p(u_2|h=1) \, J(0, u_2, 1) \quad ,$$

$$\left. \pi^1_T \sum_{u_2} p(u_2|h=0) \, J(1, u_2, 0) + (1-\pi^1_T) \sum_{u_2} p(u_2|h=1) \, J(1, u_2, 1) \right\} \quad (3\text{-}9)$$

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$$V_t^{1T}(\pi_t^1) = \min \left\{ \min_{u_1 \in \{0,1\}} \left[\pi_t^1 \sum_{u_2} p(u_2|h=0) J(u_1, u_2, 0) \right. \right.$$

$$+ (1-\pi_t^1) \sum_{u_2} p(u_2|h=1) J(u_1, u_2, 1) \Big]$$

$$c + E_{y_1(t+1)} \left\{ V_{t+1}^{1T} \left(\frac{\pi_t^1 \, p(y_1(t+1)|h=0)}{\pi_t^1 \, p(y_1(t+1)|h=0) + (1-\pi_t^1) \, p(y_1(t+1)|h=1)} \right) \right\} \right\}$$

$$(3-10)$$

Similar properties and equations hold for $V_t^{2T}(\pi_t^2)$. $\boxed{X}$

The results of Theorem 3.1 and Lemma 3.1 will be used to solve the infinite horizon decentralized Wald problem.

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SECTION 4

SOLUTION OF PROBLEM (P)

Problem (P) is the infinite horizon version of the problem investigated in [5]. Thus, in order to solve it, it is sufficient to study the behavior and properties of the value functions $V_t^{iT}\left(\pi_t^1\right)$ i = 1,2) as T → ∞. Let T → ∞. Since the set of stopping times τ_i {τ_i < T} increases with T it follows that for all t

$$V_t^{iT}(\pi) < V_t^{i(t+1)}(\pi) \quad i = 1,2 \quad .$$

Consequently the following limit is well defined:

$$V_t^i(\pi) \; \underline{\Delta} \; \lim_{T\to\infty} V_t^{iT}(\pi) \; \underline{\Delta} \; \mathrm{Inf}_T \; V_t^{iT}(\pi) = V^i(\pi) \qquad (4\text{-}1)$$

It is possible to show that the value functions $V^i(\pi)$ have the following properties:

Lemma 4.1

The value function $V^1(\pi)$ is a nonnegative concave function of π which satisfies the equation

$$V^1(\pi^1) = \min \left\{ \min_{u_1 \in \{0,1\}} \left\{ \pi^1 \sum_{u_2} p\left(u_2|h=0\right) J\left(u_1, u_2, 0\right) \right. \right.$$
$$+ \left(1-\pi^1\right) \sum_{u_2} p\left(u_2|h=1\right) J\left(u_1, u_2, 1\right) \right\} ,$$
$$\left. c + E_{y_1} \left\{ V^1 \left(\frac{\pi^1 \, p\left(y_1|h=0\right)}{\pi^1 \, p\left(y_1|h=0\right) + \left(1-\pi^1\right) \, p\left(y_1|h=1\right)} \right) \right\} \right\} \qquad (4\text{-}2).$$

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Moreover, $E_{y_1}\left\{V^1 \cdot\left(\dfrac{\pi^1 \, p(y_1|h=0)}{\pi^1 \, p(y_1|h=0) + (1-\pi^1) \, p(y_1|h=1)}\right)\right\}$ is a nonnegative concave

function of π^1 and the following inequality holds at both $\pi^1 = 0$ and $\pi^1 = 1$

$$\min_{u_1 \in \{0,1\}}\left\{\pi^1 \sum_{u_2} p(u_2|h=0) \, J(u_1, u_2, 0) + (1-\pi^1) \sum_{u_2} p(u_2|h=1) \, J(u_1, u_2, 1)\right\}$$

$$< c + E_{y_1}\left\{V^1\left(\dfrac{\pi^1 \, p(y_1|h=0)}{\pi^1 \, p(y_1|h=0) + (1-\pi^1) \, p(y_1|h=1)}\right)\right\}$$

$$(4-3)$$

Similar properties and inequalities hold for $V^2(\pi^2)$.

Proof: The proof that $E_{y_1}\left\{V^1\left(\dfrac{\pi^1 \, p(y_1|h=0)}{\pi^1 \, p(y_1|h=0) + (1-\pi^1) \, p(y_1|h=1)}\right)\right\}$ is a

nonnegative concave function as well as the inequalities at $\pi^1 = 0$, $\pi^1 = 1$

follow directly from Lemma 3.2 of [5]. $V^1(\pi)$ then is nonnegative and concave

as the minimum of nonnegative concave functions QED.

In Eq. 4-2 the term

$$\min_{u_1 \in \{0,1\}}\left\{\pi^1 \sum_{u_2} p(u_2|h=0) \, J(u_1, u_2, 0) + (1-\pi^1) \sum_{u_2} p(u_2|h=1) \, J(u_1, u_2, 1)\right\}$$

describes the cost incurred by stopping at a certain time and the term

$$c + E_{y_1}\left\{V^1\left(\dfrac{\pi^1 \, p(y_1|h=0)}{\pi^1 \, p(y_1|h=0) + (1-\pi^1) \, p(y_1|h=1)}\right)\right\}$$

describes the cost incurred by continuing to take observations at that time.

This fact and Lemma 4.1 can be used to prove the main result of this report

which is given by the following theorem:

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Theorem 4.1

The member-by-member optimal strategies of problem (P) are described by thresholds $(\alpha^1, \beta^1, \alpha^2, \beta^2)$. The thresholds of the two detectors are coupled and their computation requires the solution of the following set of nonlinear algebraic equations:

$$\alpha^1 \sum_{u_2} p(u_2|h=0)\, J(1, u_2, 0) + (1-\alpha^1) \sum_{u_2} p(u_2|h=1)\, J(1, u_2, 1)$$

$$= c + E_{y_1} \left\{ V^1 \left(\frac{\alpha^1\, p(y_1|h=0)}{\alpha^1\, p(y_1|h=0) + (1-\alpha^1)\, p(y_1|h=1)} \right) \right\} \tag{4-4}$$

$$\beta^1 \sum_{u_2} p(u_2|h=0)\, J(0, u_2, 0) + (1-\beta^1) \sum_{u_2} p(u_2|h=1)\, J(0, u_2, 1)$$

$$= c + E_{y_1} \left\{ V^1 \left(\frac{\beta^1\, p(y_1|h=0)}{\beta^1\, p(y_1|h=0) + (1-\beta^1)\, p(y_1|h=1)} \right) \right\} \tag{4-5}$$

Equations similar to 4-4 and 4-5 hold for α^2, β^2, respectively.

The conditional probabilities $\pi^i \triangleq \text{Prob}(h=0|y^{it}_t)$ are sufficient statistics for decisionmaking. At any time t the action of agent i based on the statistic π^i_t can be described as follows:

1. If $\pi^i_t > \beta^i$, declare $u_i = 0$ and stop.
2. If $\alpha^i < \pi^i_t < \beta^i$, continue taking measurements.
3. If $\pi^i_t < \alpha^i$, declare $u_i = 1$ and stop.

Hence, the stopping times τ_i ($i = 1,2$) of the two agents have the following property:

$$\tau_i = \min \left\{ t: \quad \pi^i_t > \beta^i \quad \text{or} \quad \pi^i_t < \alpha^i \right\} \tag{4-6}$$

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Proof: The threshold property of the m.b.m.o. solutions follows directly from Lemma 4.1. Equations 4-4 and 4-5 follow from Eqs. 3-2, 3-3 and 4-1. The action an agent has to take at any time t as well as the property 4-6 of the stopping times τ_i follow from Eq. 4-2 and the interpretation of each one of the terms of the right-hand side of Eq. 4-2. QED

The threshold properties of the m.b.m.o. solutions resulting from Lemma 4.1 and Theorem 4.1 are shown in Fig. 4-1. The coupling of the thresholds results because of the presence of terms $p(u_2|h=0)$ and $p(u_2|h=1)$ which are functions of (α^2, β^2). In general, $p(u_2|h=0)$ and $p(u_2|h=1)$ are very complicated functions of (α^2, β^2) and that is why it is very difficult to solve Eqs. 4-4 through 4-5 and their counterparts for α^2, β^2 numerically. Thus, the results of this note describe only the qualitative properties of the m.b.m.o. solutions of the infinite horizon decentralized Wald problem. Nevertheless, the qualitative properties of the m.b.m.o. solutions can be used to guide the design of simple suboptimal algorithms which are easy to implement. Such an algorithm will be described in [12].
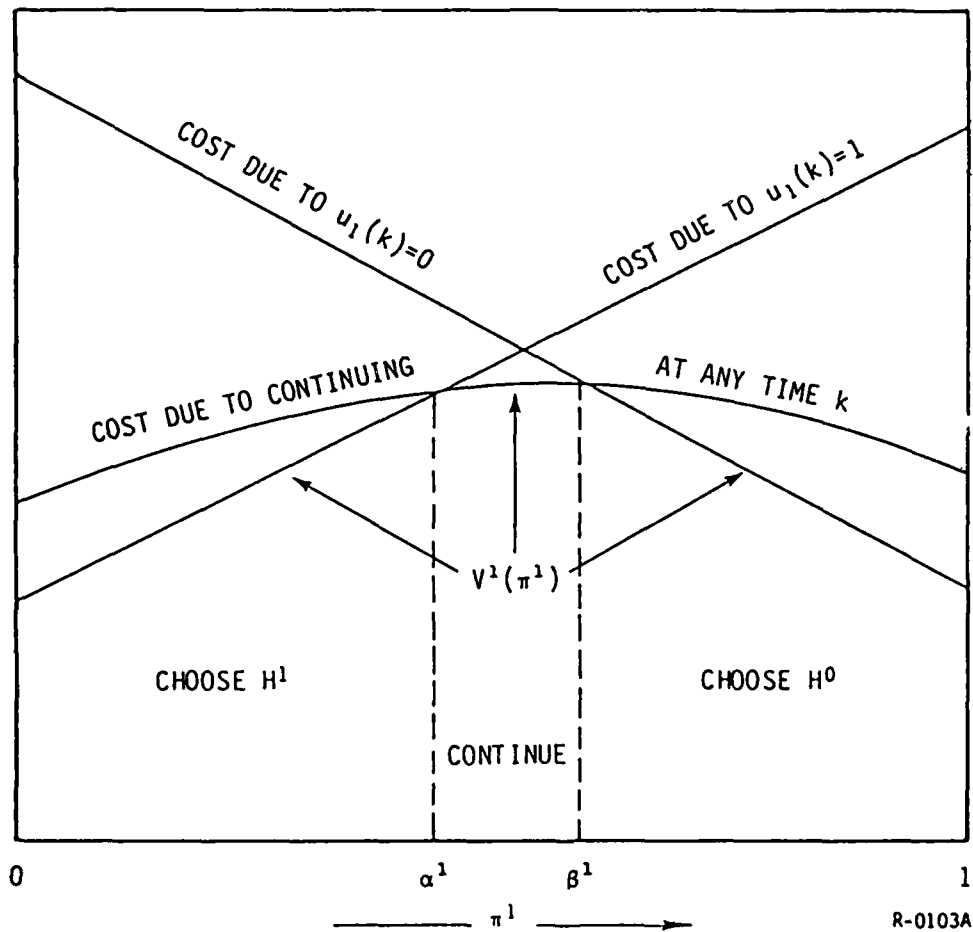
Figure 4-1. The Threshold Property of the m.b.m.o Solutions

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SECTION 5

CONCLUSIONS

The results presented in this report extend the results of [5] to the
infinite horizon decentralized Wald problem. Even though the infinite horizon
results in m.b.m.o. solutions which are described by stationary thresholds,
it is still very difficult to compute these thresholds because the coupling
of the agents through the cost results in extremely complicated dynamic pro-
gramming equations for the two agents. Thus, the design of suboptimal algo-
rithms which take advantage of the qualitative properties of the m.b.m.o. and
are easy to implement is necessary.

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APPENDIX C

TR-203

SUBOPTIMAL DECISION RULES FOR TWO SEQUENTIAL DISTRIBUTED DETECTION PROBLEMS

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TR-203

SUBOPTIMAL DECISION RULES FOR TWO SEQUENTIAL
DISTRIBUTED DETECTION PROBLEMS

By

D. Teneketzis
S. Basile
N.R. Sandell, Jr.

June 1984

ALPHATECH Inc.
2 Burlington Executive Center
111 Middlesex Turnpike
Burlington, Massachusetts 01803
(617) 273-3388

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ABSTRACT

Two Sequential Distributed Detection Problems are considered: (1) The Infinite Horizon Decentralized Wald Problem and (2) a problem with communicating detectors and ordered stopping times. The qualitative properties of the optimal solution of the Infinite Horizon Decentralized Wald problem are known, but the computation of the optimal solution is very difficult. Sequential Distributed Detection problems with communication are in general not sequentially decomposable, therefore very difficult to solve.

Suboptimal algorithms are proposed for the solution of the problems above. These algorithms combine the qualitative features of the optimal solution of the Infinite Horizon Decentralized Wald problem and results from Sequential Analysis. They are simple, easy to implement and the numerical results obtained are intuitive.

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CONTENTS

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SECTION 1

INTRODUCTION

Distributed estimation and detection problems have recently received considerable attention [1]-[3], [5]-[12}. The interest in these problems was sparked by large-scale systems such as surveillance systems, power systems, etc., where there is no centralization of information.

The distributed detection problems studied so far can be classified into (1) static problems without communication [5]-[7], (2) static problems with communication, and (3) sequential problems without communication [1]-[3], [8].

The distributed sequential detection problems solved in [1]-[3], [8} are hypothesis testing problems where there are two or more detectors who have access to different information and have to detect the correct hypothesis quickly and accurately. The detectors do not communicate but they all have a common objective, thus they are coupled only through their cost function. Even in this simple case it has been shown ([1]-[3], [8]) that the coupling through the cost causes considerable complexity in the computation of the optimal stopping rules of the detectors. More specifically, it has been shown that the optimal decision rules are characterized by thresholds whose computation requires the solution of a set of coupled dynamic programming equations. Numerical solution of these equations is in general very difficult, hence

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*References are indicated by numbers in square brackets, the list appears at the end of the main body of this report.

simple suboptimal solutions which exploit the qualitative features of the optimal solutions are needed.

The purpose of this report is to describe a suboptimal solution for two sequential distributed detection problems.

1. The infinite horizon decentralized Wald problem; and

2. A distributed sequential hypothesis testing problem with communication.

The report is organized as follows: The sequential distributed detection problems mentioned above are formulated in Section 2 and their basic features are discussed in Section 3. The proposed suboptimal algorithms for their solution appear in Section 4. Numerical results for the infinite horizon decentralized Wald problem appear in Section 5. The report concludes with a summary and suggestions for further research which appear in Section 6.

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SECTION 2

PROBLEM FORMULATION

Consider a hypothesis testing problem where one of two events is true, i.e., if h is the event to be identified, then $h \in \{0,1\}$.

Consider two detectors who observe the event in a noisy environment; the observations y^i of detector i ($i = 1,2$) are described by

$$y^i(t) = h + v^i(t) \tag{2-1}$$

It is assumed that $\{v^i(t)\}$ ($i = 1,2$) are independent identically distributed sequences which are independent of each other and independent of the event h. Thus, conditioned on the event, the observations of the two detectors are independent. The noise sequences $\{v^i(t)\}$, $i = 1,2$ are zero mean normal random variables with variance σ. It is further assumed that the observation at time $t = 1$ is free for both detectors, but each additional obsevation is costly. Let C be the cost of each additional observation after $t = 1$.

Denote by u_i the decision of detector i and let t_i be the time this decision was made. If h is the true hypothesis, then the cost incurred by the decisions u_1, u_2 is assumed to be

$$J(u_1, u_2, h) = \begin{cases} 0 & \text{if } u_1 = u_2 = h \\ 1 & \text{if } u_1 \neq u_2 \\ k & \text{if } u_1 = u_2 \neq h \ , \ k > 1 \ . \end{cases} \tag{2-2}$$

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Under these _basic assumptions_ (and some additional ones which are character-
istic to each problem) the infinite horizon decentralized Wald problem and
the sequential distributed detection problem with communication and ordered
stopping times can be formulated as follows:

2.1 THE INFINITE HORIZON DECENTRALIZED WALD PROBLEM

Consider that the two detectors do not communicate; thus, each detector's
decision is based only on the data available to it.

Then, under all the above assumptions the decentralized Wald Problem is

$$\underset{\gamma_1 \epsilon \Gamma_1, \ \gamma_2 \epsilon \Gamma_2}{\text{Minimize}} \quad E\left\{ c\tau_1 + c\tau_2 + J(u_1, u_2, h) \right\} \qquad (2\text{-}3)$$

where τ_i is the stopping time of detector i

$$\tau_i = \gamma_i(y^{i\tau_i}) \qquad\qquad (2\text{-}4)$$

and

$$y^{i\tau_i} : \ = \left(y^i(1), \ y^i(2), \dots, y^i(\tau^i) \right) \qquad (2\text{-}5)$$

$\left. \right\} \ (Q_1)$

2.2 A SEQUENTIAL DISTRIBUTED DETECTION PROBLEM WITH COMMUNICATION AND ORDERED STOPPING TIMES

Consider that detector 1 makes a decision first and communicates this
decision to detector 2. Then, based on its information and the message
received by detector 1, detector 2 makes a decision at some time.

Under the basic assumptions and the assumption above a sequential detec-
tion problem with communication and ordered stopping times can be formulated
as follows:

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Minimize $\quad E\left\{c\tau_1 + c\tau_2 + J(u_1, u_2, H)\right\}$ $\qquad$ (2-6)
$\gamma_1 \epsilon \Gamma_1, \ \gamma_2 \epsilon \Gamma_2$

where τ_i is the stopping time of detector i

$$\tau_1 = \gamma_1\left(y^{1\tau_1}\right) \qquad (2-7)$$

$$\tau_2 = \gamma_2\left(y^{2\tau_2}, u_1, \tau_1\right) \qquad (2-8)$$

$$\tau_2 > \tau_1$$
$$(2-9)$$

(Q_2)

and $y^{i\tau_i}$ is defined as in Eq. 2-5.

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SECTION 3

DISCUSSION

Since $k > 1$ the cost $J(u_1, u_2, h)$ is in general not separable (i.e., $J(u_1, u_2, h) \neq J(u_1, h) + J(u_2, h))$, consequently in both problems the decisions of the two detectors are coupled.

The absence of communication between the detector makes the decentralized Wald problem a sequential static team problem. Thus, the member-by-member optimal decision rules of the detectors can be determined by backward induction. It was shown in [3] that the member-by-member optimal decision rules are characterized by thresholds whose computation requires the solution of two coupled sets of dynamic programming equations. Numerical solution of these equations is very difficult, thus a simple suboptimal solution is desirable.

When the detectors communicate then the problem of determining the optimal stopping rules becomes much harder, because in this case the information of one detector depends on the stopping rule used by the other thus resulting in a dynamic team problem which is not sequentially decomposable. That is why good suboptimal solutions are very valuable.

In the next section a simple suboptimal solution to the problems above will be presented. The proposed solutions combine the features of the member-by-member optimal solutions of the decentralized Wald problem with some standard results from statistical sequential analysis [4].

SECTION 4

A SUBOPTIMAL SOLUTION FOR THE PROPOSED PROBLEMS

4.1 A SUBOPTIMAL SOLUTION FOR THE INFINITE HORIZON DECENTRALIZED WALD PROBLEM

The basic idea of the proposed suboptimal solution is the following.

Let $\alpha_1(\beta_1)$ be the probability of error of type 1 for detector 1(2) (i.e., the probability that if h = 0 detector 1(2) will declare h = 1); similarly, let $\alpha_2(\beta_2)$ be the probability of error of type 2 for detector 1(2) (i.e., the probability that if h = 1 is true detector 1(2) will declare h = 0). We shall write the cost (Eq. 2-4) as a function of these four quantities and then we shall minimize that cost jointly over α_1, α_2, β_1, β_2. After α_1, α_2, β_1, β_2 are determined, standard results from statistical sequential analysis will be used to determine the thresholds for the two detectors and afterward the solution of problem (Q_1) will be determined graphically.

From statistical sequential analysis ([4]) it is known that the average number of observations required to reach a decision with errors α_1 and α_2 is approximately

$$\overline{n}_1(0) = -2\sigma \left[\alpha_1 \log \frac{1 - \alpha_2}{\alpha_1} + (1 - \alpha_1)\log \frac{\alpha_2}{1 - \alpha_1} \right] \qquad (4-1)$$

when the event h = 0 is true, and

$$\overline{n}_1(1) = 2\sigma \left[(1 - \alpha_2)\log \frac{1 - \alpha_2}{\alpha_1} + \alpha_2 \log \frac{\alpha_2}{1 - \alpha_1} \right] \qquad (4-2)$$

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when the event $h = 1$ is true.

Relations similar to Eqs. 4-1 and 4-2 hold for detector 2 with β_1 and β_2 in place of α_1 and α_2, respectively. Using Eqs. 4-1, 4-2 and 2-3 we can approximately write the cost to be minimized as

$$E \left\{ c\tau_1 + c\tau_2 + J(u_1, u_2, h) \right\}$$

$$= 2\sigma c(1-p) \left[(1 - \alpha_2)\log \frac{1 - \alpha_2}{\alpha_1} + \alpha_2 \log \frac{\alpha_2}{1 - \alpha_1} + (1 - \beta_2)\log \frac{1 - \beta_2}{\beta_1} \right.$$

$$\left. + \beta_2 \log \frac{\beta_2}{1 - \beta_1} \right] - 2\sigma c p \left[\alpha_1 \log \frac{1 - \alpha_2}{\alpha_1} + (1 - \alpha_1)\log \frac{\alpha_2}{1 - \alpha_1} \right.$$

$$\left. + \beta_1 \log \frac{1 - \beta_2}{\beta_1} + (1 - \beta_1)\log \frac{\beta_2}{1 - \beta_1} \right] + (1 - \alpha_2)\beta_2 (1 - p)$$

$$+ \alpha_2(1 - \beta_2) (1 - p) + \alpha_1(1 - \beta_1)p + (1 - \alpha_1) \beta_1 p + k \alpha_1\beta_1 p$$

$$+ k \alpha_2\beta_2(1 - p) : = f(\alpha_1, \alpha_2, \beta_1, \beta_2) \qquad (4\text{-}3)$$

The minimization of $f(\alpha_1, \alpha_2, \beta_1, \beta_2)$ over these four parameters gives their optimal values. Then the relations

$$A_1 = \log \frac{1 - \alpha_2}{\alpha_1} \qquad (4\text{-}4)$$

$$A_2 = \log \frac{\alpha_2}{1 - \alpha_1} \qquad (4\text{-}5)$$

$$B_1 = \log \frac{1 - \beta_2}{\beta_1} \tag{4-6}$$

$$B_2 = \log \frac{\beta_2}{1 - \beta_1} \tag{4-7}$$

from standard sequential analysis ([4]) can be used to determine the thresholds of the detectors. After the thresholds are specified by Eqs. 4-4 through 4-7 the decisions of the two detectors can be graphically determined by Fig. 4-1.

At any time t, the sum of observations $\sum_t y^1(t)$ up to that time is a sufficient statistic for decisionmaking for detector 1. As long as the sum $\sum_t y^1(t)$ remains between the two parallel lines ℓ_1, ℓ_2 (Fig. 4-1) detector 1 continues to take measurements. The first instant of time the sum $\sum_t y^1(t)$ is above ℓ_1 or below ℓ_2 detector 1 stops and accepts $h = 1$ if $\sum_t y^1(t)$ is above ℓ_1 and $h = 0$ if $\sum_t y^1(t)$ lies below ℓ_2. Similar results holds for detector 2.

The proposed algorithm is suboptimal because the expressions 4-1 and 4-2 for the average number of measurements are only approximate; the exact computation of the average number of measurements is very complicated. Even though suboptimal, the algorithm proposed above captures some of the basic features of the optimal solution of the infinite horizon decentralized Wald problem; namely, the decision rules of the two detectors are described by thresholds and the thresholds are given in terms of the probabilities of error of type 1 and type 2 which are determined by joint optimization for the two detectors. The joint optimization is very simple as it only requires the minimization of Eq. 4-3 with respect to α_1, α_2, β_1, β_2. As it will be seen in Section 5, the numerical results obtained by the algorithm are intuitively appealing.
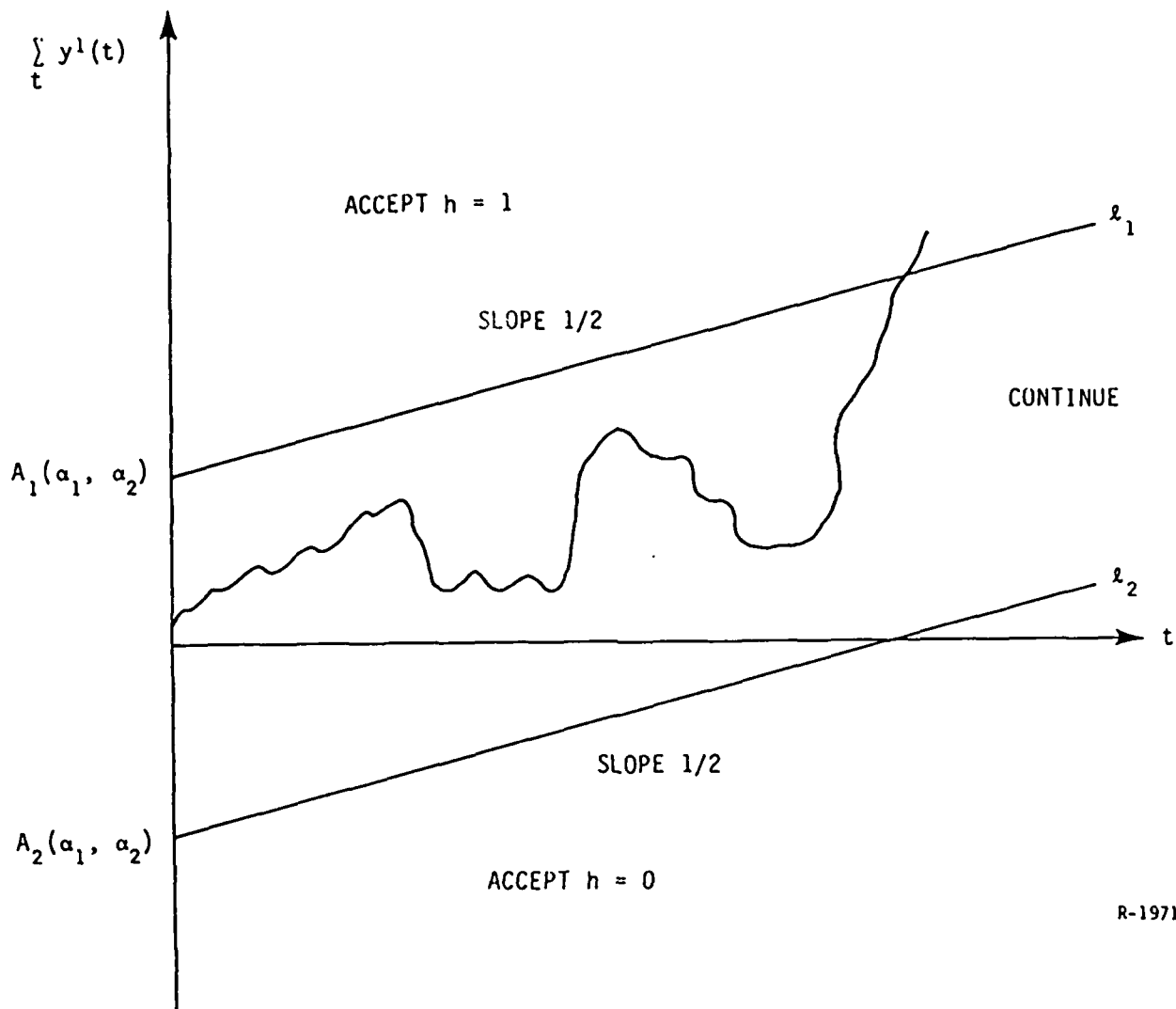
Figure 4-1. Detector 1's Decision Rule

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4.2 A SUBOPTIMAL SOLUTION FOR A SEQUENTIAL DISTRIBUTED DETECTION PROBLEM WITH COMMUNICATION AND ORDERED STOPPING TIMES

The basic idea of the suboptimal solution for the problem (Q_2) of Section 2 is the following:

Let α_1, α_2, β_1, β_2, be the probabilities of error of type 1 and type 2 for detector 1 and detector 2, respectively. Following the procedure of subsection 4.1 we can determine (off-line) first the probabilities α_1, α_2, β_1, β_2, which minimize the cost (Eq. 4-3) and afterwards the thresholds $A(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$ $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ using Eq. 4-4 through 4-7. Then, the deci- of the two detectors can be graphically determined as follows: Detector 1's decision is determined in exactly the same manner as in subsection 4.1 (Fig. 4-1). Detector 2 uses detector 1's decision as well as the time that decision was made and treats them as an additional observation in the following manner:

If the decision of detector 1 is $u_1 = 1$ and is made at $t = \tau_1$, this means that

$$\sum_{t=1}^{\tau_1} y^1(t) > A_1(\alpha_1, \alpha_2) + \frac{\tau_1}{2} \quad ; \qquad (4\text{-}8)$$

then, detector 2 treats the decision $u_1 = 1$ at τ_1 as an additional measurement with value $A(\alpha_1, \alpha_2) + \frac{1}{2} \tau_1$, adds it to his previous measurements and proceeds from that point on to make a decision in the same way as in subsection 4.1.

If the decision of detector 1 is $u_1 = 0$ and is made at $t = s_1$, this means that

$$\sum_{t=1}^{s_1} y^1(t) < A_2(\alpha_1, \alpha_2) + \frac{s_1}{2} \quad ; \qquad (4\text{-}9)$$

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then, detector 2 treats the decision $u_1 = 0$ at s_1 as an additional measurement with value $- \left(A(\alpha_1, \alpha_2) + \dfrac{s_1}{2} \right)$, adds it to the previous measurements and proceeds from that point to make a decision in the same way as in subsection 4.1 (Fig. 4-1).

The suboptimal algorithm proposed here is graphically depicted in Fig. 4-2.

In the suboptimal algorithm proposed above, the interpretation of the first detector's message by the second detector is very easy because it is assumed that the stopping rules used by the detectors are arbitrarily fixed and known beforehand to both of them. When the optimal solution to problem (Q_2) is sought, then the interpretation of the message of detector 1 depends on the stopping rule used by that detector; since the stopping rule has to be determined, the interpretation of the message by the second detector is a highly nontrivial task and leads to optimization problems which are not sequentially decomposable. Since it is in general very difficult to solve such problems, simple suboptimal solutions like the one proposed above must be investigated and evaluated.

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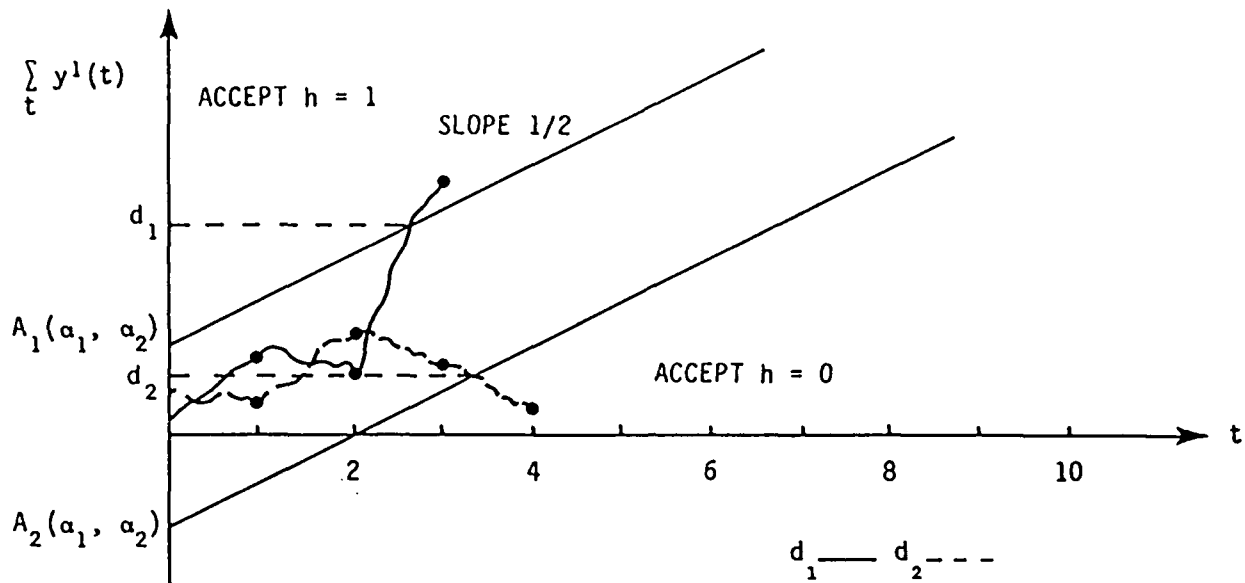
Figure 4-2a. Detector 1's Decision Rule
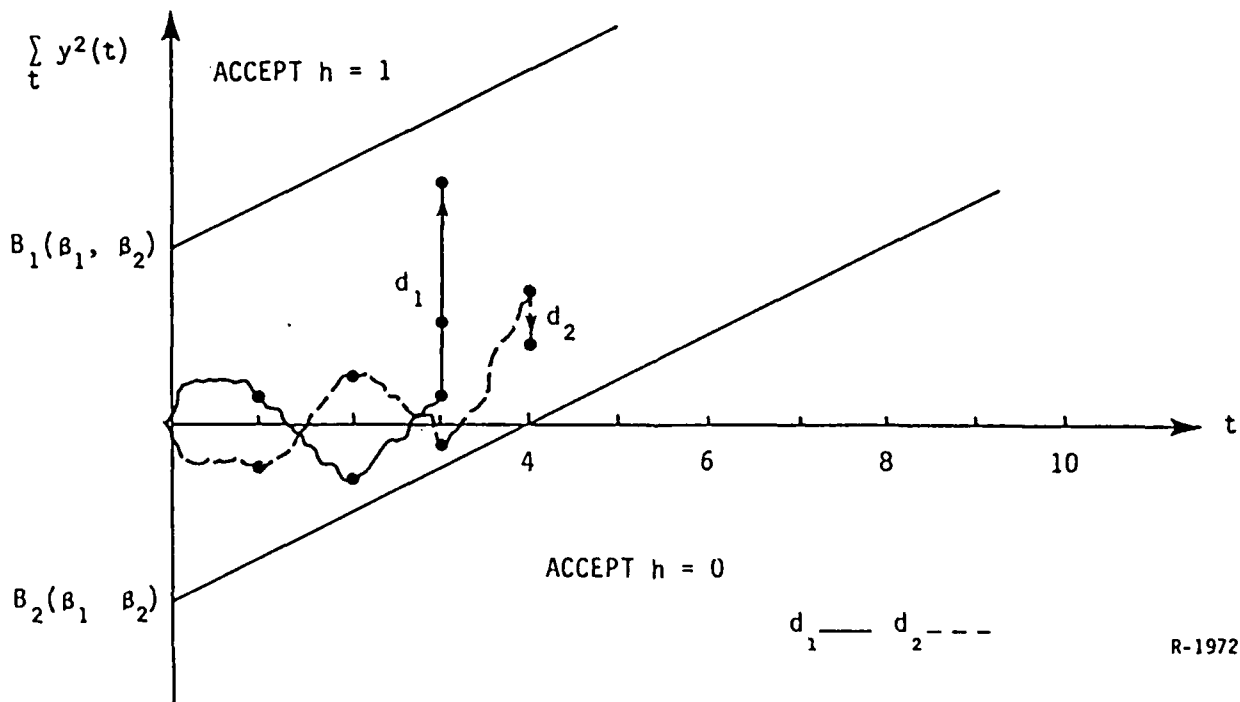
Figure 4-2b. Detector 2's Decision Rule

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SECTION 5

NUMERICAL RESULTS

In this section the numerical results obtained by the implementation of the algorithm proposed in subsection 4.1 are presented. The probabilities of error $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ as well as the thresholds $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ have been computed for various values of the following parameters:

1. The prior probability $p = \text{Prob}(h = 0)$;

2. The variance σ of the observation noise;

3. The cost c of the observations;

4. The penalty k arising when both detectors' decisions are *wrong*.

We shall present each one of our parametric studies separately, and we shall interpret the results obtained by these studies. Note that the cost (4-3) is a nonconvex function of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$, thus the values of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ determined by the minimization of Eq. 4-3, correspond to local minima. These minima depend on the initial guess of $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ used in the minimization of Eq. 4-3; as it will be shown below different initial guesses give rise to different solutions for $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$. Some of the local minima of Eq. 4-3 result in $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Such local minima are obtained when the minimization of Eq. 4-3 is initiated with $\alpha_1^{in} = \alpha_2^{in} = \beta_1^{in} = \beta_2^{in}$. The numerical results we present below correspond to minima for which $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$.

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5.1 THE VARIATION OF α_1, α_2, β_1, β_2, $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$, AS A FUNCTION OF p, k

Figures 5-1 through 5-3 present the variation of the probabilities of error of type 1 and 2 as well as the variation of the thresholds as functions of the prior proability p for fixed c, k, σ. It is expected that as p, the prior probability of h = 0, increases, the likelihood that the final decision will be h = 1 decreases. Using the results of [3] we can justify this claim as follows:

The final decision of detector i is h = 1 only if

$$p(h = 0|y^{it}) < \ell^i_1 \qquad \text{at time t and}$$

$$\ell^i_1 < p(h = 0|y^{it}) < \ell^i_2 \qquad \text{for } 1 < s < t \quad,$$

where ℓ^i_1 and ℓ^i_2 are the thresholds determined in [3]. As the prior probability p of h = 0 increases the probability of the sest of measurements y^{it} that could cause $p(h = 0|y^{it})$ to drop below ℓ^i_1 decreases, thus decreasing the probability of error of type 1. On the contrary, as p increases, the probability of the set of measurements y^{it} that would result in $p(h = 0|y^{it}) < \ell^i_2$ increases, consequently the probability of error of type 2 increases. This is indeed shown in Figs. 5-1 and 5-2.

The thresholds $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ are also expected to vary with p as follows: As p increases each detector would be biased more and more towards declaring h = 0. Consequently, as p increases, the area where h = 0 is accepted in Fig. 4-1 would increase and the area where h = 1 is accepted in Fig. 4-1 would decrease; hence, the thresholds

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$A_1(\alpha_1, \alpha_2)$ $\left(B_1(\beta_1, \beta_2)\right)$ and $A_2(\alpha_1, \alpha_2')$ $\left(B_2(\beta_1, \beta_2)\right)$ defined by Eqs. 4-4 through 4-7, should both increase as p increases. This is indeed shown in Fig. 5-3. The results of Figs. 5-1 through 5-3 assume that k = 4, c = 0.1, σ = 0.5. When the cost of observations and the noise variance in the observations remain unchanged but the terminal cost k due to two errors changes, then it is expected that the probabilities α_1, α_2, β_1, β_2 as well as the thresholds $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ will change.

More specifically, it is intuitively expected that as k increases the error probabilities α_1, α_2, β_1, β_2 will decrease for a fixed p. Such a behavior is expected for the following reason: As the penalty due to errors increases, the detectors tend to become more conservative and more cautious, thus they tend to base their decisions on more reliable information; consequently, the probability of error is reduced. This can be verified by comparing Figs. 5-1 and 5-2 with Figs. 5-4 and 5-5. It can be seen from this comparison that for the same p, c, σ,

$$\alpha_1(k = 6) < \alpha_1(k = 4)$$

$$\alpha_2(k = 6) < \alpha_2(k = 4) \quad .$$

A continuous variation of the probabilities of error versus k is shown in Fig. 5-6. (Some of the local minima result in $\alpha_1 = \alpha_1 = \beta_1 = \beta_2$ when p = 0.5 and the minimization of Eq. 4-3 is initiated with $\alpha_1^{in} = \alpha_2^{in} = \beta_1^{in} = \beta_2^{in}$. Thus, Fig. 5-6 describes the variation of α_1, α_2, β_1, β_2 as a function of k).

As far as the behavior of the thresholds $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$ and $B_2(\beta_1, \beta_2)$ as a function of k is concerned, we can expect the following:

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Since as k increases, the two detectors become more conservative and more cautious, the areas of Fig. 4-1 where h = 0 and h = 1 are accepted should get smaller. Consequently, as k increases, the upper threshold $A_1(\alpha_1, \alpha_2)$ $(B_1(\beta_1, \beta_2))$ should increase whereas the lower threshold $A_2(\alpha_1, \alpha_2)$ $(B_2(\beta_1, B_2))$ should decrease. This behavior can be verified by comparing Figs. 5-3 and 5-7. The continuous variation of the thresholds versus k is shown in Fig. 5-8.

5.2 THE VARIATION OF α_1, α_2, β_1, β_2, $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ AS A FUNCTION OF c, σ

Figures 5-9 and 5-10 present the variation of the probabilities of error of type 1 and 2 as well as the variation of the thresholds as a function of the observation cost c for fixed p, k, σ. We set p = 0.5; then some of the local minima result in $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$ when the minimization of Eq. 4-3 is initiated with $\alpha_1^{in} = \alpha_2^{in} = \beta_1^{in} = \beta_2^{in}$; thus, Fig. 5-9 describes the variation of α_1, α_2, β_1, β_2 as a function of c. It is intuitively expected that as the cost of observations increases the detectors would tend to take less observations, therefore, the quality of information upon which their decisions are based would get worse with increasing c, consequently the probabilities α_1, α_2, β_1, β_2 would increase. This behavior is actually shown in Fig. 5-9. The variation of the thresholds with c is expected to be the following: As c increases the detectors would tend to make a final decision more quickly, consequently the areas where h = 0 and h = 1 are accepted in Fig. 4-1 will get larger with increasing c. Hence it is expected that as c increases, the lower

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threshold $A_2(\alpha_1, \alpha_2)$ $\left(B_2(\beta_1, \beta_2)\right)$ will increase whereas the upper threshold $A_1(\alpha_1, \alpha_2)$ $\left(B_1(\beta_1, \beta_2)\right)$ will decrease. This behavior is indeed shown in Fig. 5-10.

Figures 5-11 and 5-12 present the variation of the probabilities of error of type 1 and type 2 as well as the variation of the thresholds as a function of the noise variance σ for fixed p, k, c. We set p = 0.5; then some of the local minima result in $\alpha_1 = \alpha_2 = \beta_1 = \beta_2$ when $\alpha_1^{in} = \alpha_2^{in} = \beta_1^{in} = \beta_2^{in}$. Such minima are shown in Figs. 5-11 and 5-12. It is intuitively expected that as the noise variance σ increases the quality of information of the detectors gets worse, hence the probabilities of error α_1, α_2, β_1, β_2 increase. This behavior is actually shown in Fig. 5-11. Notice that for $\sigma > 20$ the probabilities of error α_1, α_2, β_1, β_2 approach the value 0.5, hence for $\sigma > 20$ the observations provide the detectors with very little additional information, thus the observations are practically useless.

As far as the behavior of the thresholds is concerned, we can expect the following: As the quality of observations gets worse, the detectors would tend to rely more and more on their prior information, thus they would tend to make decisions more quickly. Consequently as σ increases, the areas where h = 0 and h = 1 are accepted would tend to increase; hence, the upper threshold $A_1(\alpha_1, \alpha_2)$ $\left(B_1(\beta_1, \beta_2)\right)$ will decrease and the lower threshold $A_2(\alpha_1, \alpha_2)$ $\left(B_2(\beta_1, \beta_2)\right)$ will increase with increasing σ. This behavior of the thresholds is shown in Fig. 5-12. As noted before, for $\sigma > 20$ the observations are practically useless, therefore the thresholds $A_1(\alpha_1, \alpha_2)$ $\left(B_1(\beta_1, \beta_2)\right)$ and $A_2(\alpha_1, \alpha_2)$ $\left(B_2(\beta_1, \beta_2)\right)$ approach very close to each other because the detectors make decisions based practically on their prior information.

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So far the results presented in this section correspond to local minima for which $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$. There are many local minima (Eq. 4-3) other than the symmetric ones. Below we present such a solution.

5.3 A NONSYMMETRIC SOLUTION

When the initial values of α_1, α_2, β_1, β_2 used for the minimization of Eq. 4-3 are such that $\alpha_1^{in} = \beta_1^{in} = \alpha_2^{in} = \beta_2^{in}$ then the symmetric local minima of subsections 5.1 and 5.2 result. However, if the initial values of α_1, α_2, β_1, β_2 used in the minimization of Eq. 4-3 are $\alpha_1^{in} \neq \alpha_2^{in} \neq \beta_1^{in} \neq \beta_2^{in}$ then the resulting local minimum are not symmetric. An example of such a local minimum is given below.

For $p = 0.9$, $k = 4$, $c = 0.05$ and initial values

$$\alpha_1^{in} = 0.2 \qquad \alpha_2^{in} = 0.5 \qquad \beta_1^{in} = 0.74 \qquad \beta_2^{in} = 0.3$$

the resulting local minimum is

$$\alpha_1 = 0.000164 \qquad \alpha_2 = 0.999457$$

$$\alpha_1 = 0.1150 \qquad \beta_2 = 0.5694$$

and the thresholds $A_1(\alpha_1, \alpha_2)$, $A_2(\alpha_1, \alpha_2)$, $B_1(\beta_1, \beta_2)$, $B_2(\beta_1, \beta_2)$ are

$$A_1 = 1.1972429 \qquad A_2 = -0.0003791$$

$$B_1 = 2.791737 \qquad B_2 = -0.4410045 \quad .$$

It is expected that the behavior of the nonsymmetric solutions as a function of p, k, c, σ will be qualitatively the same as that of the symmetric solutions.

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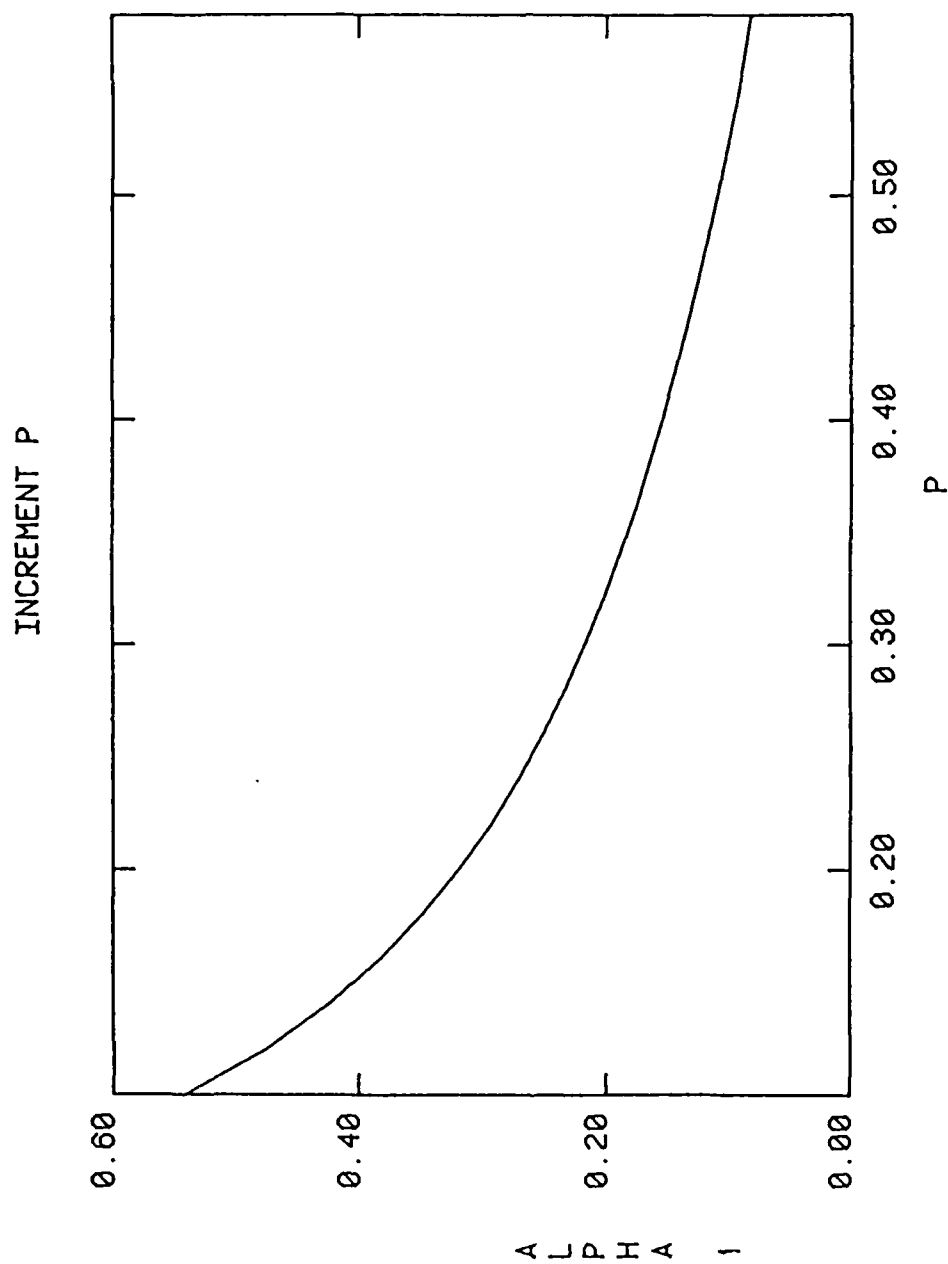
Figure 5-1. Type 1 Error versus p (k = 4, c = 0.1, σ = 0.5)

Figure 5-2. Type 2 Error versus p ($k = 4$, $c = 0.1$, $\sigma = 0.5$)
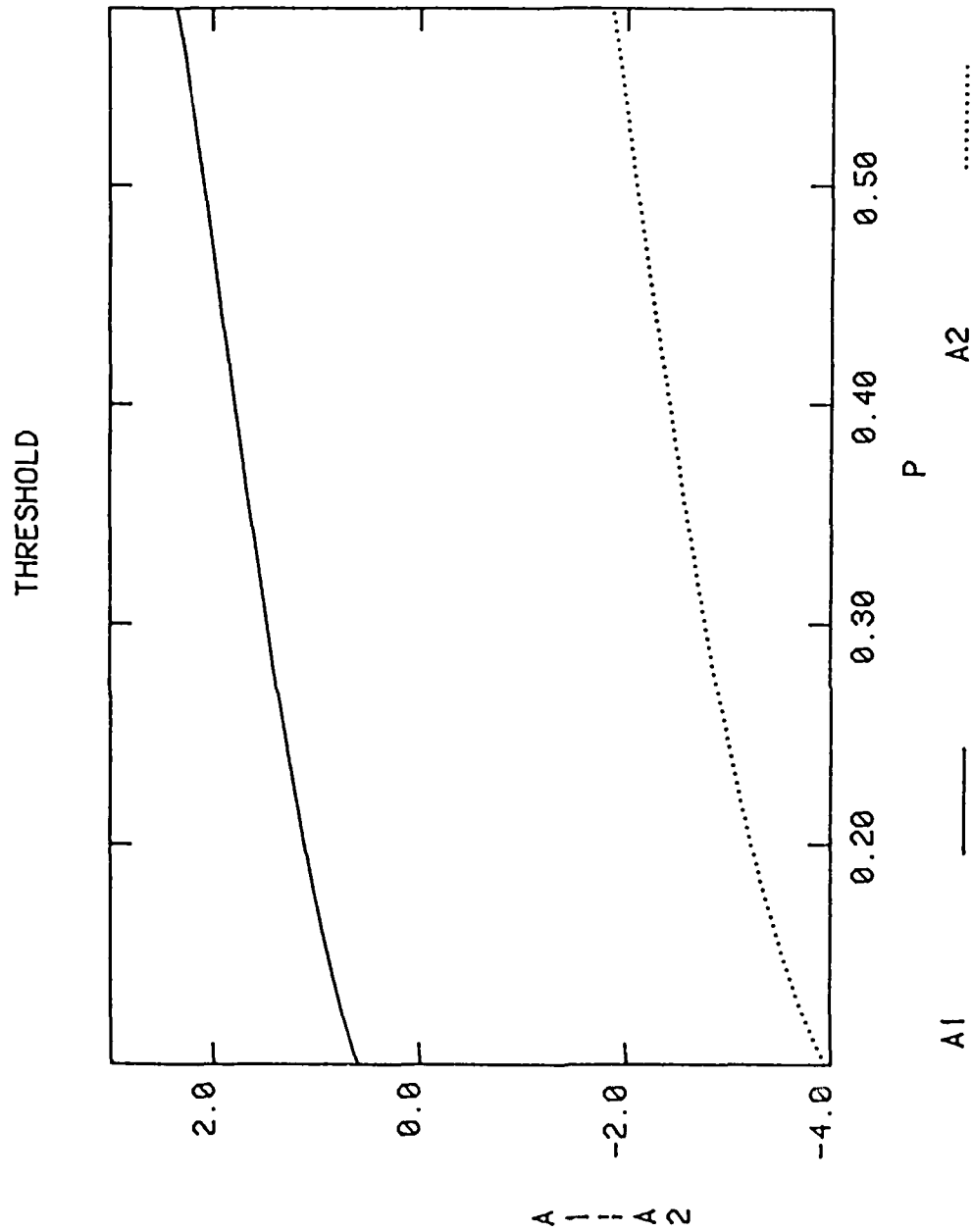
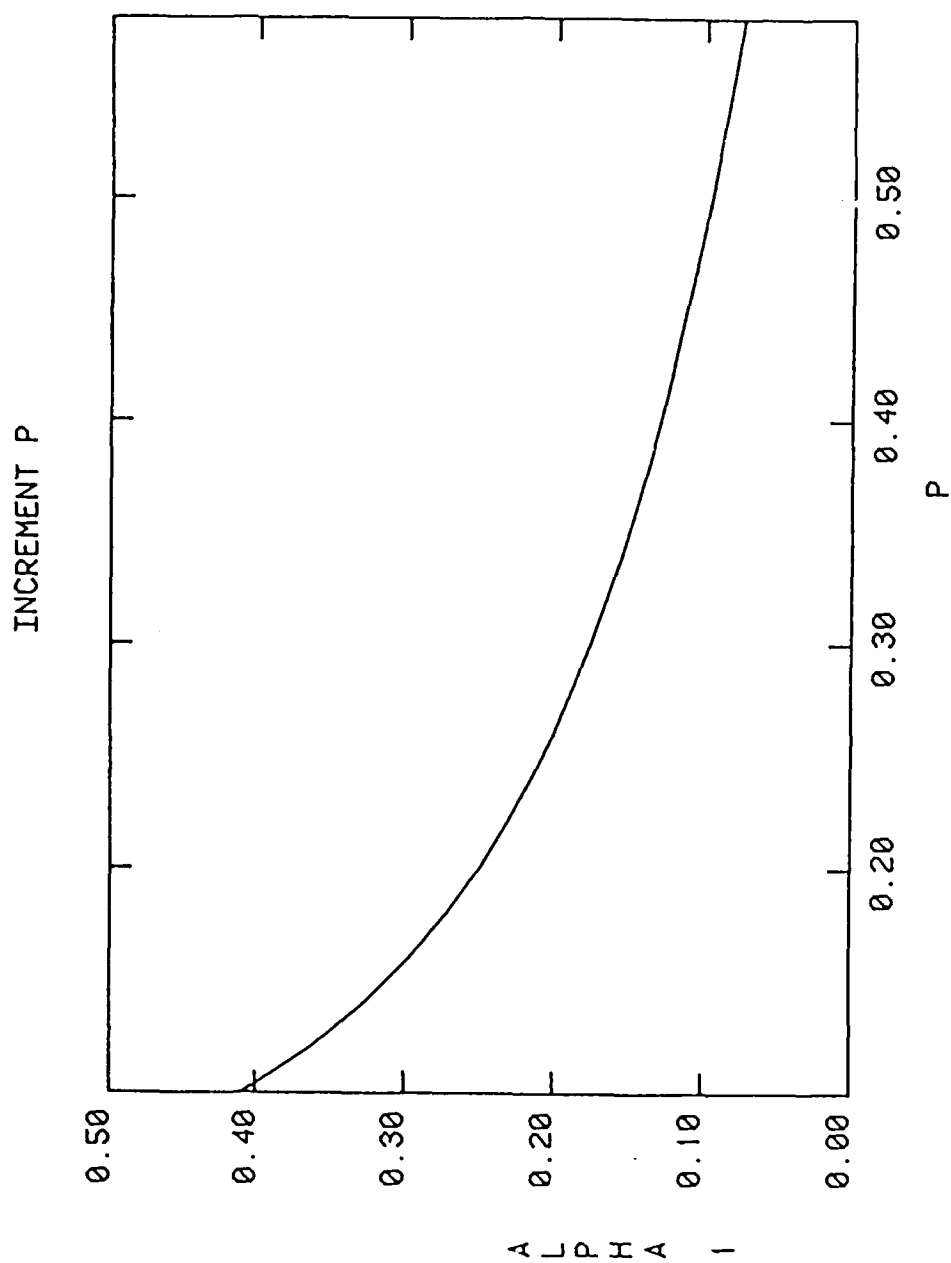THRESHOLD

Figure 5-3. Thresholds versus p (k = 4, c = 0.1, σ = 0.5)
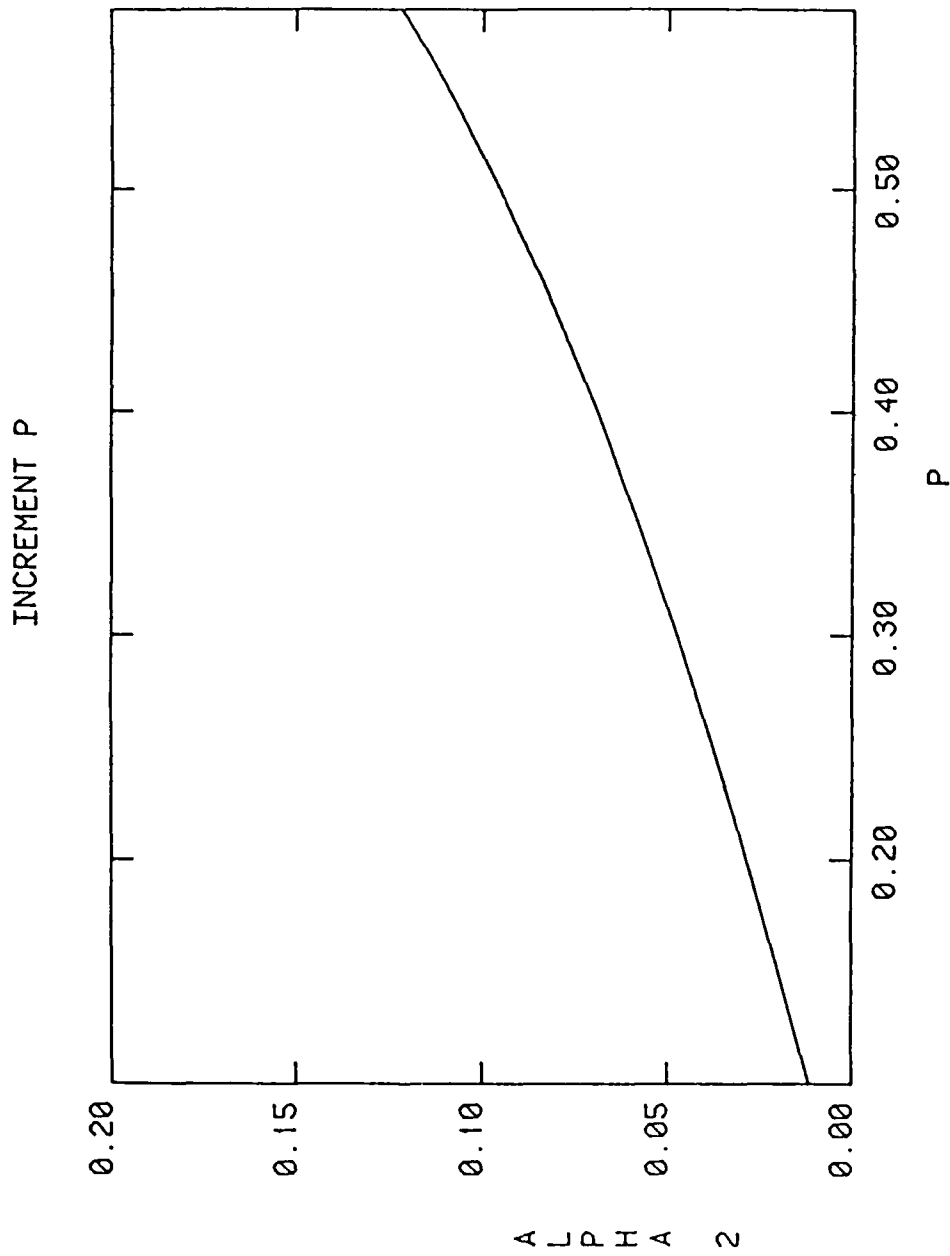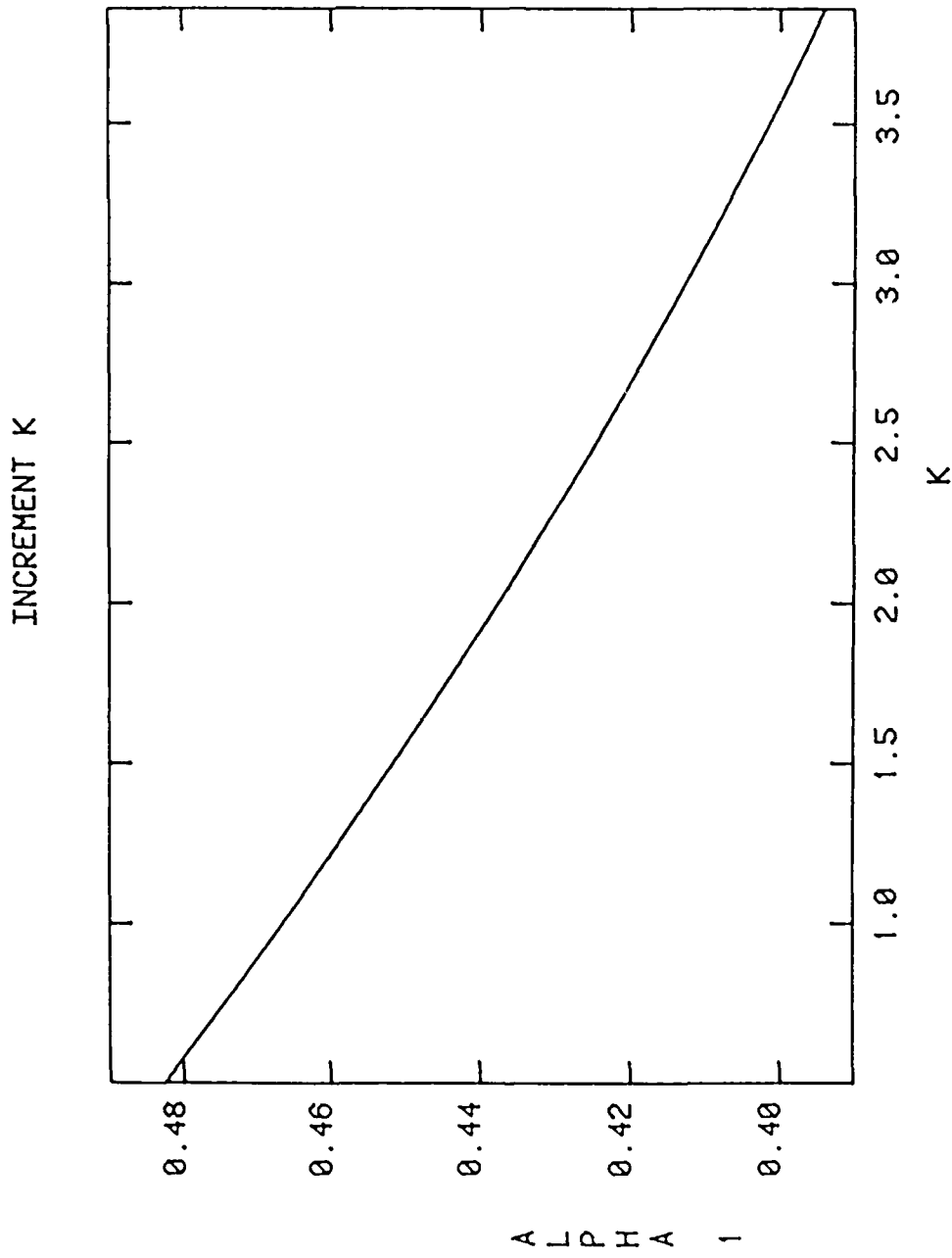
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Figure 5-4. Type 1 Error versus p (k = 6, c = 0.1, σ = 0.5)

Figure 5-5. Type 2 Error versus p (k = 6, c = 0.1, σ = 0.5)

Figure 5-6. Type 1 Error versus k (p = 0.5, c = 0.99, σ = 0.5)
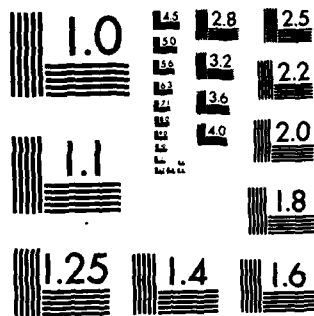
MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A
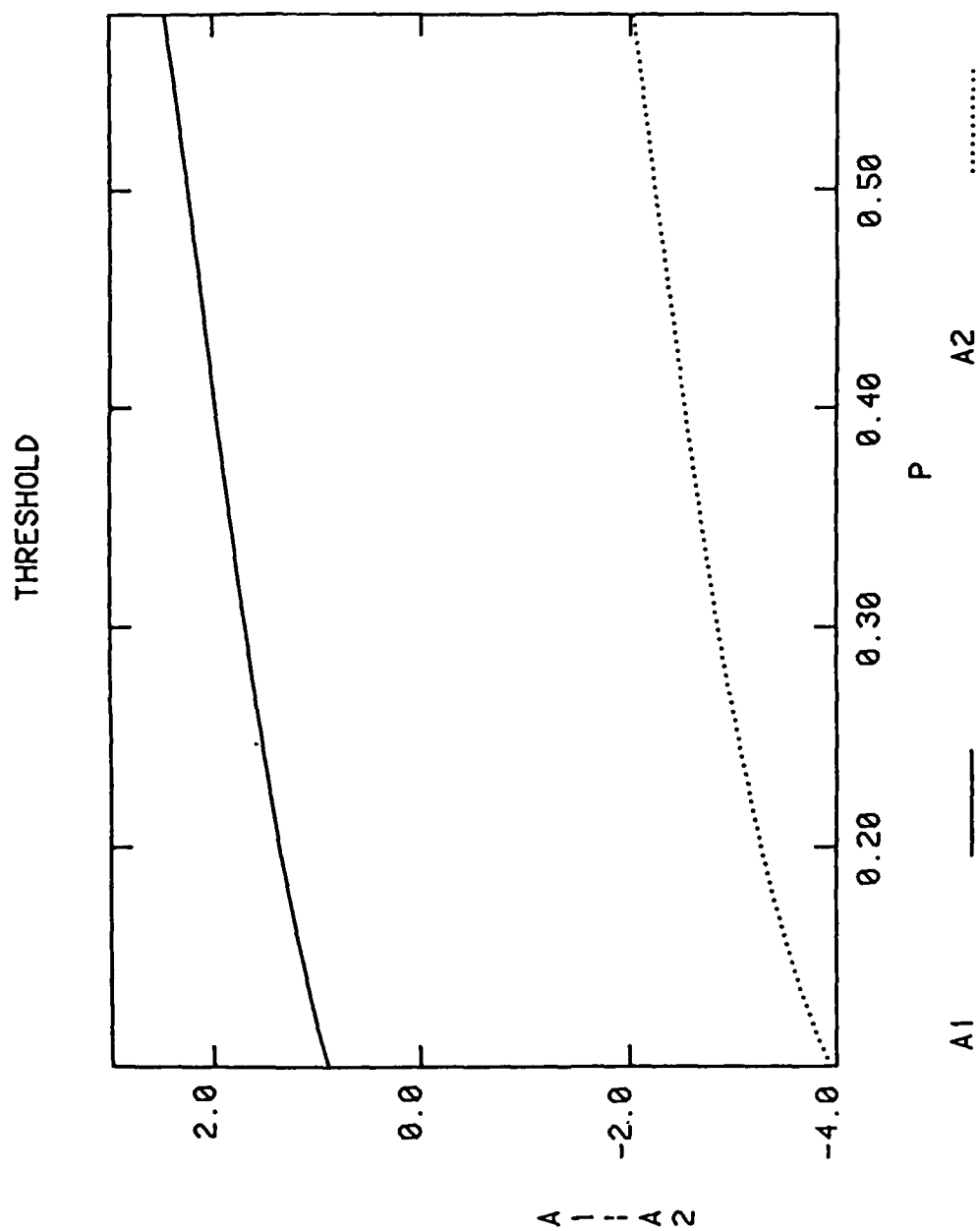
Figure 5-7. Thresholds versus p ($k = 6$, $c = 0.1$, $\sigma = 0.5$)
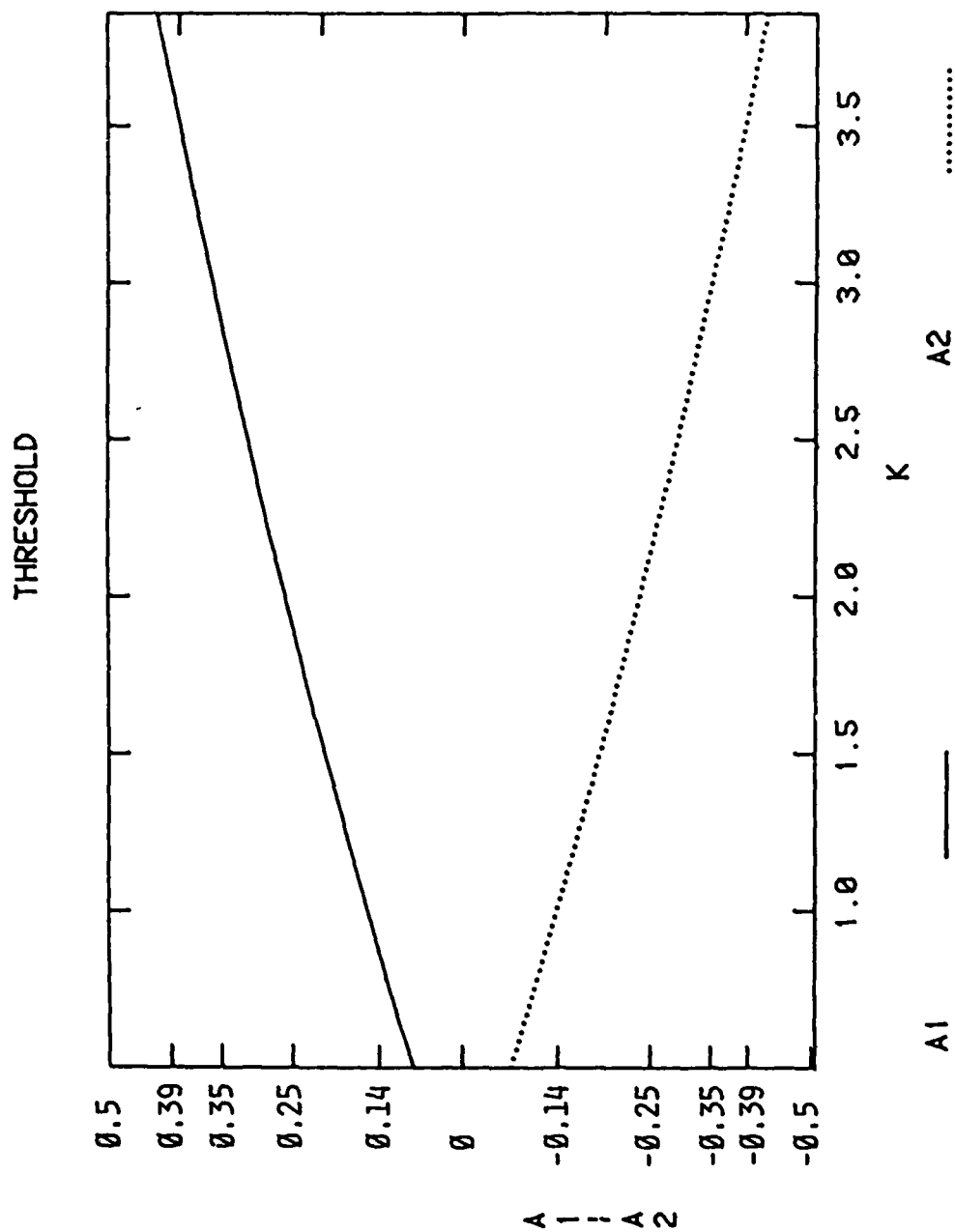
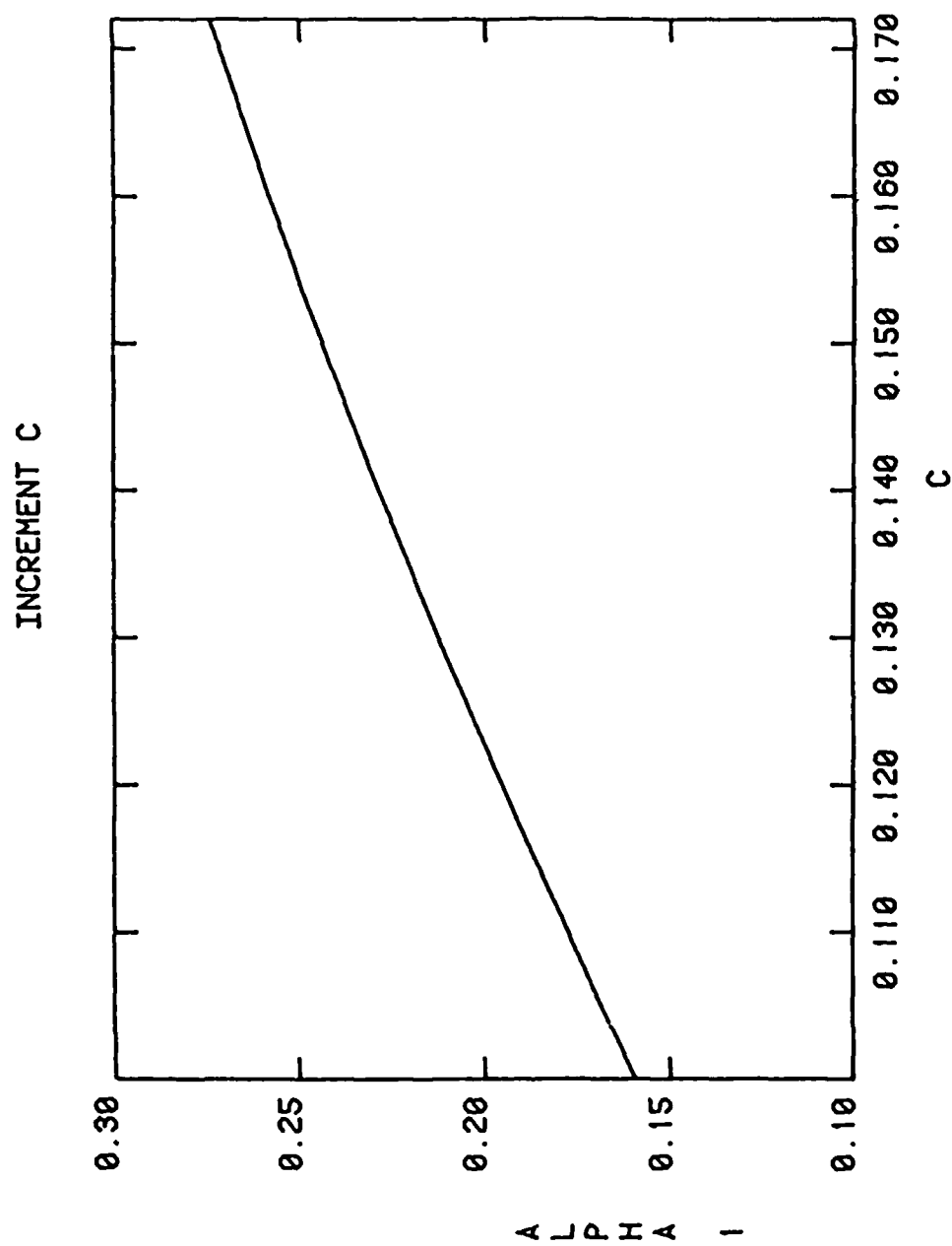Figure 5-8. Thresholds versus k (p = 0.5, c = 0.99, σ = 0.5)

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Figure 5-9. Type 1 Error versus c (p = 0.5, k = 1, σ = 0.5)

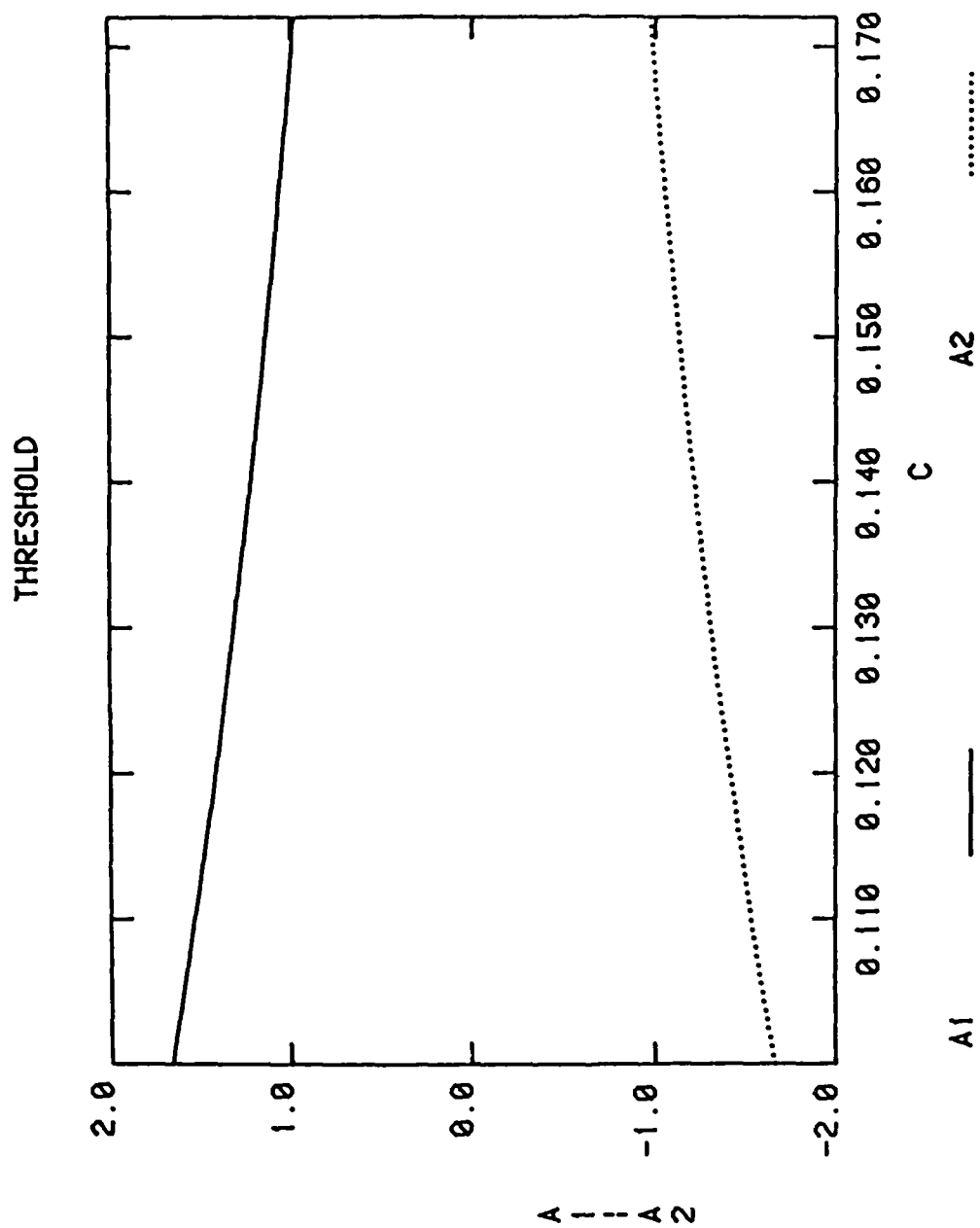THRESHOLD

Figure 5-10.   Thresholds versus c (p = 0.5, k = 1, σ = 0.5)

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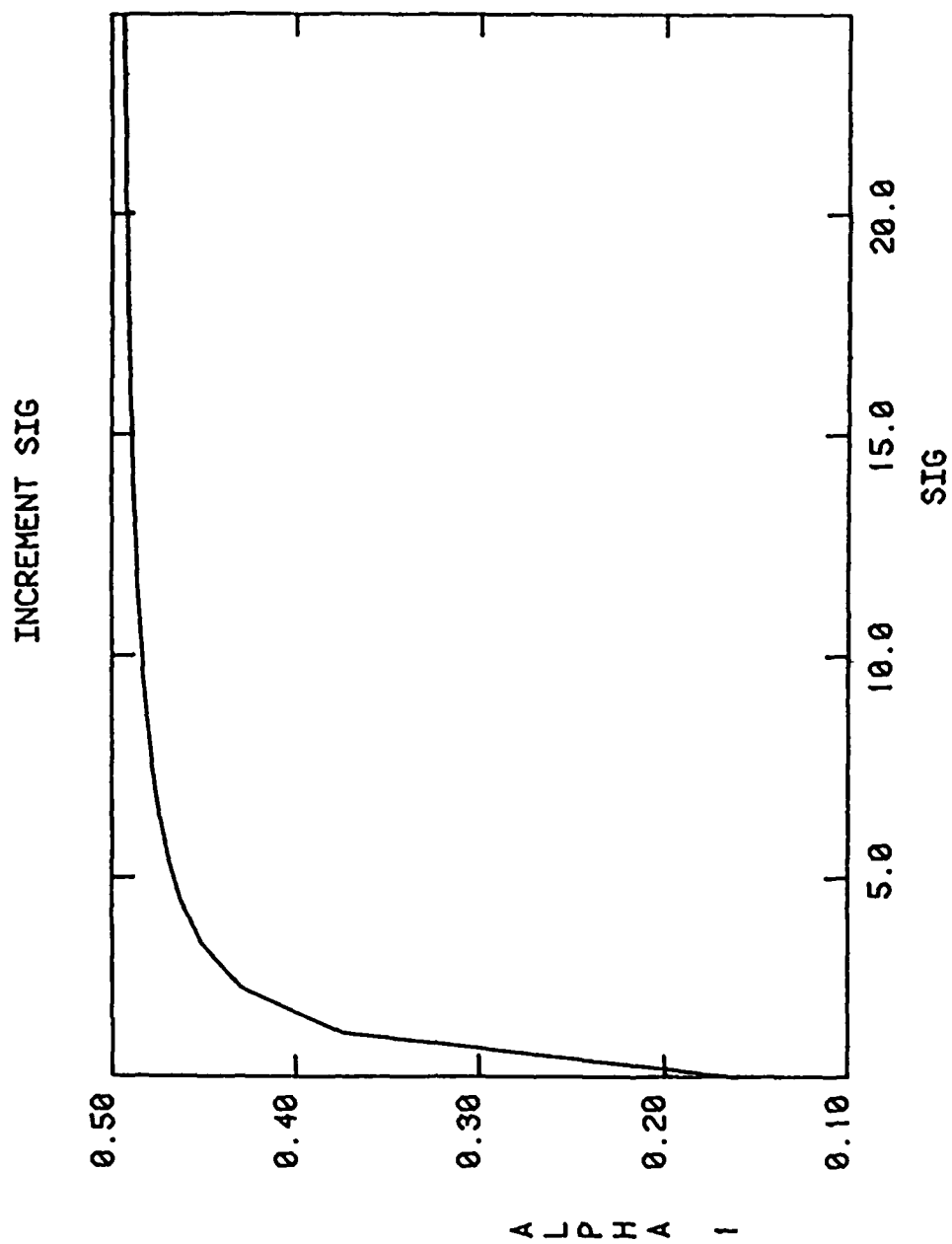
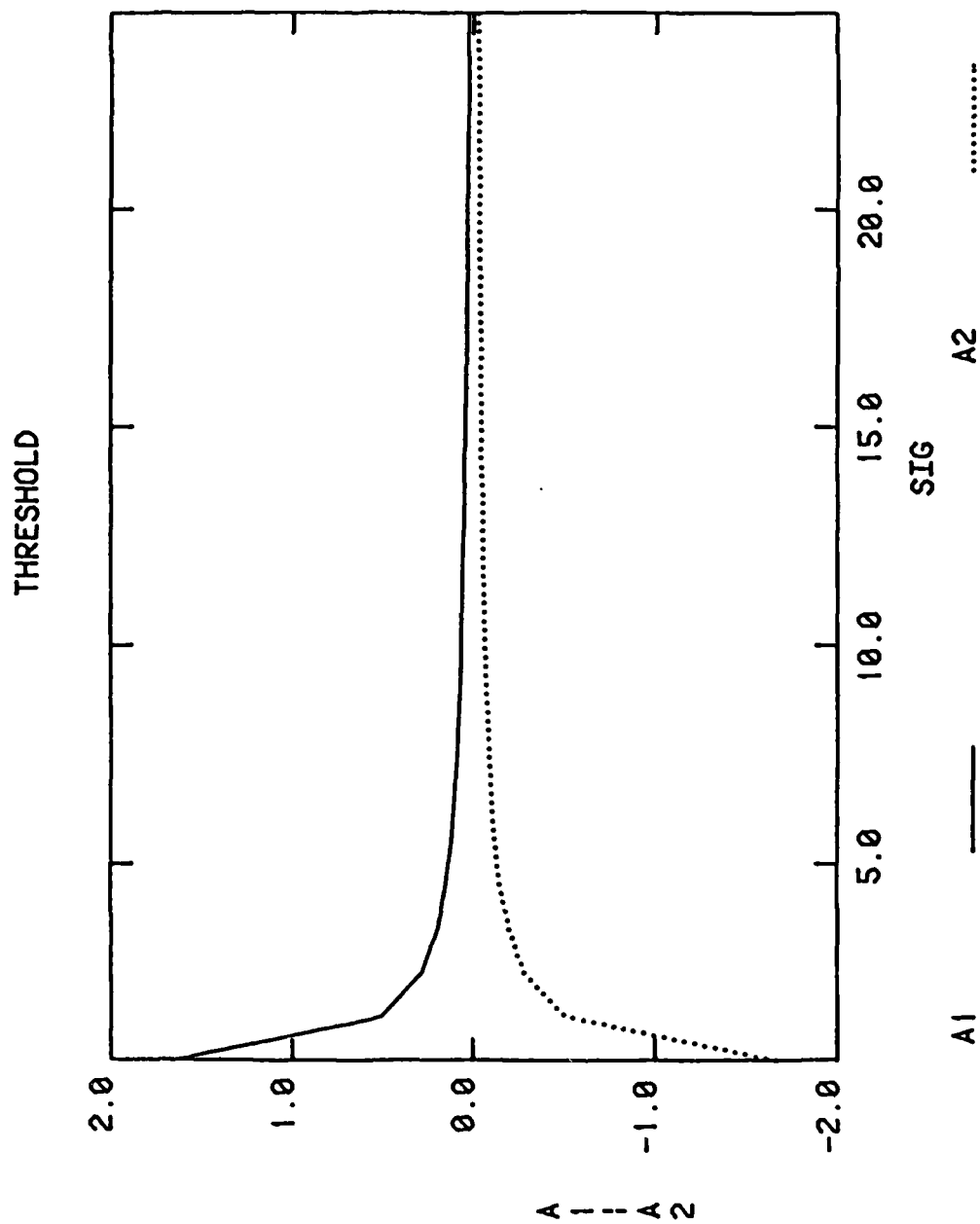Figure 5-11. Type 1 Error versus (p = 0.5, k = 1, c = 0.1)

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THRESHOLD

Figure 5-12. Thresholds versus σ (p = 0.5, k = 1, c = 0.1)

SECTION 6

SUMMARY - CONCLUSIONS

In this report we presented suboptimal solutions for two sequential distributed detection problems: (1) The Infinite Horizon Decentralized Wald Problem and (2) A Sequential Distributed Detection Problem with Communication and ordered stopping times.

The suboptimal algorithm proposed for the solution of the infinite horizon decentralized Wald problem captures the basic features of the optimal solution and is easy to implement. The numerical results obtained by the proposed algorithm, and presented in Section 5, are intuitively appealing.

When communication is allowed between the detectors, the resulting distributed sequential detection problems are not sequentially decomposable in general. That is why simple optimal solutions are valuable. The algorithm proposed in subsection 4.2 is simple and very easy to implement as it requires the same amount of computation as the algorithm proposed for the infinite horizon decentalized Wald problem.

The simplicity of the proposed algorithms is one of their major advantages. A performance analysis would be desirable to determine how close to the optimal these algorithms perform.

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